(1) (20 points) Prove that if \( X \subset \mathbb{R}^n \times \mathbb{R}^m \) is rectifiable and bounded, then \( X \) has measure zero if and only if the following set has measure 0 in \( \mathbb{R}^n \):
\[
\{ p \in \mathbb{R}^n \text{ so that } X \cap p \times \mathbb{R}^m \text{ does not have measure 0 in } \mathbb{R}^n \}
\]
(Hint: Prove and use the fact that if \( Y \) is bounded, \( Y \) has measure zero if \( \int \chi_Y = 0 \), and if \( Y \) has measure 0, \( \int \chi_Y = 0 \).)

(2) (30 points) Compute the volume of the \( n \) dimensional unit ball for all \( n \).
(See the hints in Munkres chapter 19, problem 6.)

(3) (20 points) Show that the following extended integral exists:
\[
\int_{\{0 < \|x\| < 1\} \subset \mathbb{R}^n} \|x\|^{-n}
\]

(4) (30 points) Let \( G \) be a compact Lie group with dimension greater than 0.
Use the notation \( l_g : G \rightarrow G \) to denote the action on \( G \) by left multiplication \( l_g h = gh \), and the notation \( r_g : G \rightarrow G \) to denote the action on \( G \) by right multiplication, \( r_g h = hg \).
(a) Prove that there exists a unique density \( \sigma_G \) on \( G \) so that
\[
\int_G \sigma_G = 1
\]
and \( \sigma_G \) is preserved by left multiplication:
\[
l_g^* \sigma_G = \sigma_G \quad \forall g \in G
\]
(b) Show that this density is also preserved by right multiplication:
\[
r_g^* \sigma_G = \sigma_G \quad \forall g \in G
\]
(c) A smooth action of \( G \) on a manifold \( M \) is a family of diffeomorphisms \( \phi_g : M \rightarrow M \)
so that the map
\[
(g, m) \mapsto \phi_g(m) : G \times M \rightarrow M
\]
is smooth, and
\[
\phi_g \circ \phi_h = \phi_{gh}
\]
Prove (using some kind of averaging procedure) that if \( \phi \) is a smooth action of a compact Lie group \( G \) on \( M \) that there exists a positive density \( \sigma \) on \( M \) which is preserved by the action of \( G \). In other words,
\[
\phi_g^* \sigma = \sigma \quad \forall g \in G
\]