HOMEWORK FOR 18.101, FALL 2007
ASSIGNMENT 6
DUE 11AM FRIDAY DECEMBER 7 IN ROOM 108

(1) (20 points) Prove that if $X \subset \mathbb{R}^n \times \mathbb{R}^m$ is rectifiable and bounded, then X has measure zero if and only if the following set has measure 0 in $\mathbb{R}^n$:

$$\{p \in \mathbb{R}^n \text{ so that } X \cap p \times \mathbb{R}^m \text{ does not have measure 0 in } \mathbb{R}^n\}$$

(Hint: Prove and use the fact that if Y is bounded, Y has measure zero if $\overline{\int} \chi_Y = 0$, and if Y has measure 0, $\underline{\int} \chi_Y = 0$.)

(2) (30 points) Compute the volume of the n dimensional unit ball for all n. (See the hints in Munkres chapter 19, problem 6.)

(3) (20 points) Show that the following extended integral exists:

$$\int_{\{0<\|x\|<1\}\subset\mathbb{R}^n} \|x\|^{1-n}$$

(4) (30 points) Let G be a compact Lie group with dimension greater than 0. Use the notation $l_g : G \longrightarrow G$ to denote the action on G by left multiplication $l_g h = gh$, and the notation $r_g : G \longrightarrow G$ to denote the action on G by right multiplication, $r_g h = hg$.

(a) Prove that there exists a unique density σ_G on G so that

$$\int_G \sigma_G = 1$$

and σ_G is preserved by left multiplication:

$$l_g^* \sigma_G = \sigma_G \qquad \forall g \in G$$

(b) Show that this density is also preserved by right multiplication:

$$r_g^* \sigma_G = \sigma_G \qquad \forall g \in G$$

(c) A smooth action of G on a manifold M is a family of diffeomorphisms

$$\phi_g : M \longrightarrow M$$

so that the map

$$(g, m) \mapsto \phi_g(m) : G \times M \longrightarrow M$$

is smooth, and

$$\phi_g \circ \phi_h = \phi_{gh}$$

Prove (using some kind of averaging procedure) that if ϕ is a smooth action of a compact Lie group G on M that there exists a positive density σ on M which is preserved by the action of G. In other words,

$$\phi_g^* \sigma = \sigma \qquad \forall g \in G$$