HOMEWORK FOR 18.101, FALL 2007
ASSIGNMENT 5
DUE 11AM WEDNESDAY NOVEMBER 21 IN ROOM 108

Note that you have a little longer to complete this homework. Start it early, because it is a little harder than usual!

(1) Give an example of a manifold M with two smooth vector fields v and w so that the partial differential equation

$$\frac{\partial w_t}{\partial t} = L_v w_t$$
$$w_0 = w$$

has more than one solution.

(2) Prove that if v is a complete C^1 vector field on M, and α is a C^1 one form, then if $L_v\alpha = 0$, the flow of v preserves α in the sense that

$$\Phi_{tv}^* \alpha = \alpha$$

(3) If $f : M \longrightarrow \mathbb{R}^k$ is a C^1 map, and M is a manifold of dimension less than k, prove that $f(M) \subset \mathbb{R}^k$ has measure 0.

(4) Let Q be the unit cube, and let P_n denote the partition of Q into cubes with side lengths of size 2^{-n}. Prove that a bounded function f is Riemann integrable on Q if and only if

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$$

(5) Use partitions of unity to prove the following:

(a) If M is a closed submanifold of $\mathbb{R}^n$ and v is a smooth vector field on M, prove that there exists some smooth vector field u on $\mathbb{R}^n$ so that u restricted to M is equal to v.

(b) If M is a submanifold of $\mathbb{R}^n$ and $f : M \longrightarrow \mathbb{R}^k$ is a smooth map, prove that there exists an open set U containing M and a smooth map $\tilde{f} : U \longrightarrow \mathbb{R}^k$ so that the restriction of $\tilde{f}$ to M is f. Show that if M is a closed submanifold of $\mathbb{R}^k$, then U can be taken to be $\mathbb{R}^n$.

(6) Given a continuous real valued function f on $\mathbb{R}^n$ and a smooth, compactly supported function g on $\mathbb{R}^n$, define the convolution of f and g to be the function

$$(f * g)(x) := \int_{\mathbb{R}^n} f(s)g(x - s)$$

(The above integral should be interpreted as an integral of the compactly supported function of s, given by $f(s)g(x - s)$.)

(a) Show that $f * g$ is continuous.

(b) Show that $f * g$ is smooth by showing that the partial derivative

$$D_i(f * g) = f * D_i g$$

(c) Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a 'bump function' satisfying the following conditions

- ϕ is smooth

- $\phi(x) = 0$ if $\|x\| \geq 1$
- $\int_{\mathbb{R}^n} \phi = 1$

Define for each $t > 0$,

$$\phi_t = t^{-n}\phi\left(\frac{x}{t}\right)$$

Prove that if f is a continuous function, then $f * \phi_t$ converges to f as $t \to 0$ uniformly on any compact subset of $\mathbb{R}^n$.

(d) Prove that if f is a continuous, real valued function on a manifold M, for any $\epsilon > 0$, there exists a smooth function g on M so that $|f(p) - g(p)| < \epsilon$ for all $p \in M$.
(Hint: Use coordinate charts and a partition of unity on M.)