HOMEWORK FOR 18.101, FALL 2007
ASSIGNMENT 4
DUE 11AM FRIDAY NOVEMBER 2 IN ROOM 108

(1) Prove, using the local existence and uniqueness theorems for ordinary differential equations, that if a C^1 vector field v on $\mathbb{R}^N$ is tangent to a closed manifold $M \subset \mathbb{R}^N$ (in other words, if $p \in M$, $v(p) \in T_pM$), then any integral curve of v which intersects M must be contained entirely inside M.

(2) Put coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ on $\mathbb{R}^{2n}$. Given a smooth function $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$, the Hamiltonian flow defined by H is the flow given by the vector field

$$v_H := \sum_i \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}$$

(a) Show that H is a conserved quantity of v_H.

(b) Show that if F is smooth, $L_{v_H} F = -L_{v_F} H$.

(c) Show that if the smooth function F is a conserved quantity of H, then $[v_H, v_F] = 0$. (This means that the flow given by H preserves v_F, and that the flows given by H and F commute.)

(3) (a) Let M be a compact manifold, and v be a smooth vector field on M. Suppose that Φ_{tv} is the one parameter group of diffeomorphisms generated by v. Define $(L_v)^n h$ for a smooth function h to be L_v applied to h n times. Prove that if there exists some number N so that $(L_v)^N h = 0$, then

$$e^{tL_v}h := \sum_{n=0}^{\infty} \frac{t^n}{n!}(L_v)^n h = (\Phi_{tv})^* h := h \circ \Phi_{tv}$$

(b) Prove the analogous formula for a smooth vector field w so that $(L_v)^N w = 0$:

$$e^{tL_v}w = (\Phi_{-tv})_* w := \Phi^*_{tv} w$$

(4) A Lie group G is a group which is also a manifold, so that the group multiplication is a smooth map.

$$m : G \times G \longrightarrow G$$

and so that the operation of taking an inverse is a diffeomorphism:

$$(\cdot)^{-1} : G \longrightarrow G$$

(A group is a set G with an identity I, a multiplication map $m : G \times G \longrightarrow G$, and an inverse map $(\cdot)^{-1} : G \longrightarrow G$ so that $m(g, g^{-1}) = I = m(g^{-1}, g)$, $m(g, I) = g = m(I, g)$, and $m(m(h, g), k) = m(h, m(g, k))$. This last property called associativity allows us to write $m(m(h, g), k)$ as hgk, saving space.)

We shall use the notation l_g to indicate the diffeomorphism

$$l_g : G \longrightarrow G$$

given by multiplication on the left. So

$$l_g(h) = m(g, h)$$

(a) Show that the following defines a smooth vector field on G: Given any vector $v(0) \in T_I G$, define

$$v(g) := T_I l_g(v(0))$$

(Hint: this is given by composing the smooth map Tm with some smooth map $G \longrightarrow TG \times TG$.)

(b) Show that the collection of such vector fields is equal to the set of smooth vector fields v which are invariant under left multiplication:

$$(l_h)_* v = v$$

The set of such vector fields is called the Lie algebra $\mathfrak{g}$ of G.

(c) Show that if $v, w \in \mathfrak{g}$, then $[v, w] \in \mathfrak{g}$.

(d) Prove that if $v \in \mathfrak{g}$, then the flow Φ_{tv} is defined for all time. Show also that it satisfies the following equation:

$$\Phi_v(g) = m(g, \Phi_v(I))$$

In other words, flowing by a left multiplication invariant vector field corresponds to multiplication on the right!