HOMEWORK FOR 18.101, FALL 2007
ASSIGNMENT 3
DUE 11AM FRIDAY OCTOBER 19 IN ROOM 108

(1) For C^2 functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, write a formula for $D_1 D_2 (f \circ g)$ in terms of the derivatives up to second order of f and g.

(2) Prove the following facts:

(a)
$$d(fg) = f dg + g df$$

(b)
$$f^* dg = d(g \circ f)$$

(c)
$$f^*(\lambda\alpha) = (\lambda \circ f) f^*(\alpha)$$

(d)
$$\alpha(f_*(v)) \circ f = f^*\alpha(v)$$

(e)
$$f_*(\lambda v) = (\lambda \circ f^{-1}) f_* v$$

Note that these formulas look simpler if you define $f_*\lambda = \lambda \circ f^{-1}$ and $f^*\lambda = \lambda \circ f$.

(3) Prove that if f is a differentiable function on a compact manifold M which is not the empty set or a single point, then there exist at least two points p_1 and p_2 on M where $df|_{p_i} = 0$.

(4) (a) Suppose that $U \subset \mathbb{R}^n$ is open, and $v := \sum v_i \frac{\partial}{\partial x_i}$, $w := \sum w_i \frac{\partial}{\partial x_i}$ are C^∞ vector fields on U. Prove that there exists a unique smooth vector field $[v, w]$ with the following property:

$$L_{[v,w]} f = L_v(L_w f) - L_w(L_v f) \text{ for all smooth } f : U \longrightarrow \mathbb{R}$$

This is called the Lie bracket of v and w. Give a formula for $[v, w]$ in terms of the derivatives of v_i and w_i.

(b) Show that the Lie bracket of any two C^∞ vector fields on a manifold is also defined, and prove that if f is a diffeomorphism

$$f_*[v, w] = [f_* v, f_* w]$$

(c) Suppose that u, v, w are smooth vector fields. Prove the following identity, called the Jacobi identity

$$[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$$