(1) For $C^2$ functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$, write a formula for $D_1 D_2 (f \circ g)$ in terms of the derivatives up to second order of $f$ and $g$.

(2) Prove the following facts:

(a) $d(fg) = f dg +gd f$

(b) $f^*dg = d(g \circ f)$

(c) $f^*(\lambda \alpha) = (\lambda \circ f)^* (\alpha)$

(d) $\alpha (f^*(v)) \circ f = f^* \alpha(v)$

(e) $f^*(\lambda v) = (\lambda \circ f^{-1}) f^* v$

Note that these formulas look simpler if you define $f^* \lambda = \lambda \circ f^{-1}$ and $f^* \lambda = \lambda \circ f$.

(3) Prove that if $f$ is a differentiable function on a compact manifold $M$ which is not the empty set or a single point, then there exist at least two points $p_1$ and $p_2$ on $M$ where $df|_{p_i} = 0$.

(4) (a) Suppose that $U \subset \mathbb{R}^n$ is open, and $v := \sum v_i \frac{\partial}{\partial x_i}$, $w := \sum w_i \frac{\partial}{\partial x_i}$ are $C^\infty$ vector fields on $U$. Prove that there exists a unique smooth vector field $[v, w]$ with the following property:

$$L_{[v,w]} f = L_v (L_w f) - L_w (L_v f)$$

This is called the Lie bracket of $v$ and $w$. Give a formula for $[v,w]$ in terms of the derivatives of $v_i$ and $w_i$.

(b) Show that the Lie bracket of any two $C^\infty$ vector fields on a manifold is also defined, and prove that if $f$ is a diffeomorphism

$$f_* [v,w] = [f_* v, f_* w]$$

(c) Suppose that $u, v, w$ are smooth vector fields. Prove the following identity, called the Jacobi identity

$$[u, [v,w]] + [w, [u,v]] + [v, [w,u]] = 0$$