HOMEWORK FOR 18.101, FALL 2007
ASSIGNMENT 2
DUE 11AM FRIDAY OCTOBER 5 IN ROOM 108

(1) (a) (10 points) Prove that the following is a 2 dimensional submanifold of $\mathbb{R}^4$:
$$\{x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0\} \cap \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4\}$$
(b) (10 points) What is the tangent space to this manifold at the point $(x_1, x_2, x_3, x_4) = (1, 1, -1, -1)$?
(c) (10 points) Prove that there exists a smooth function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ so that $g(1, -1) = (1, -1)$ and $(g_1(x_2, x_3), x_2, x_3, g_2(x_2, x_3))$ is contained inside this manifold for (x_2, x_3) sufficiently close to $(1, -1)$. Find $Dg(1, -1)$.

(2) (20 points) Prove that if $X \subset \mathbb{R}^N$ is a smooth submanifold of $\mathbb{R}^N$, then $TX \subset T\mathbb{R}^N$ is a smooth submanifold.

(3) (a) (20 points) Prove that $O(n)$, the set of $n \times n$ matrices that preserve the dot product is a compact $\frac{n(n-1)}{2}$ dimensional manifold.
(b) (10 points) If I denotes the $n \times n$ identity matrix, find $T_I O(n)$.

(4) (20 points) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a smooth map, and $Y \subset \mathbb{R}^m$ be a smooth k dimensional submanifold. We say that f is transverse to Y if whenever $f(p) \in Y$,
$$T_{f(p)}\mathbb{R}^m = T_p f(T\mathbb{R}^n) + T_{f(p)}Y$$
(In other words, if $v \in T_{f(p)}\mathbb{R}^m$, then $v = T_p f(v_1) + v_2$ for some $v_1 \in \mathbb{R}^n$, and $v_2 \in T_{f(p)}Y$.)

Prove that if f is transverse to Y, then $f^{-1}(Y) \subset \mathbb{R}^n$ is a smooth $k+n-m$ dimensional submanifold.