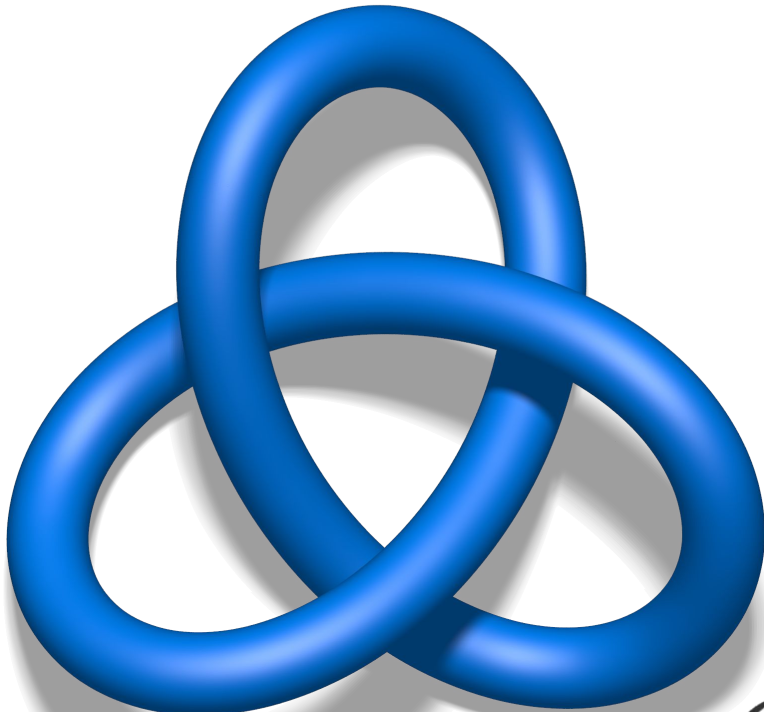
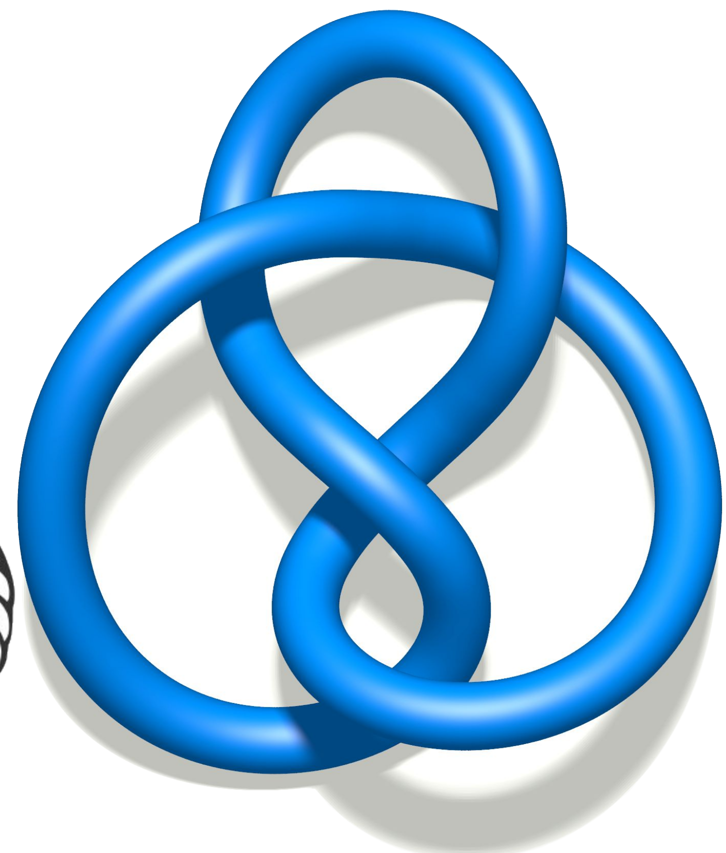

Knots and how to
detect knottiness.



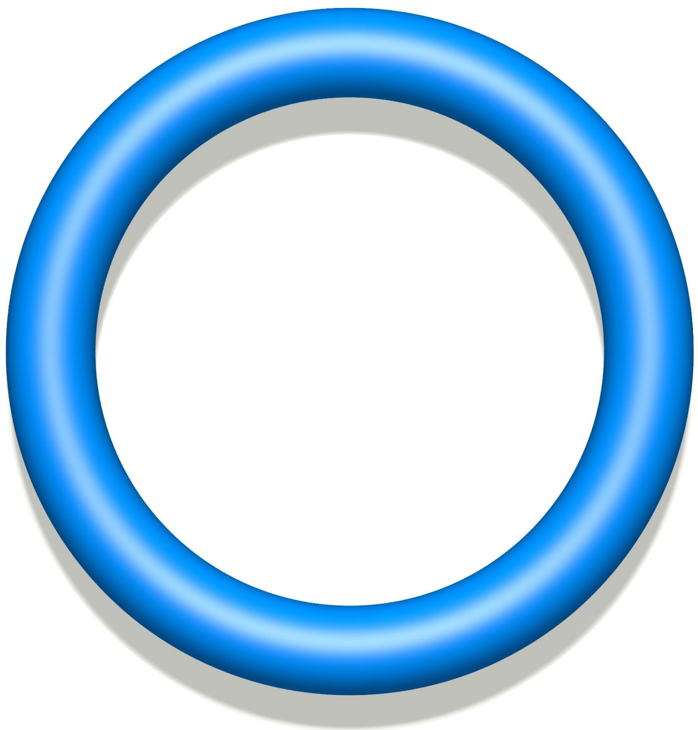
Trefoil knot

Figure-8 knot

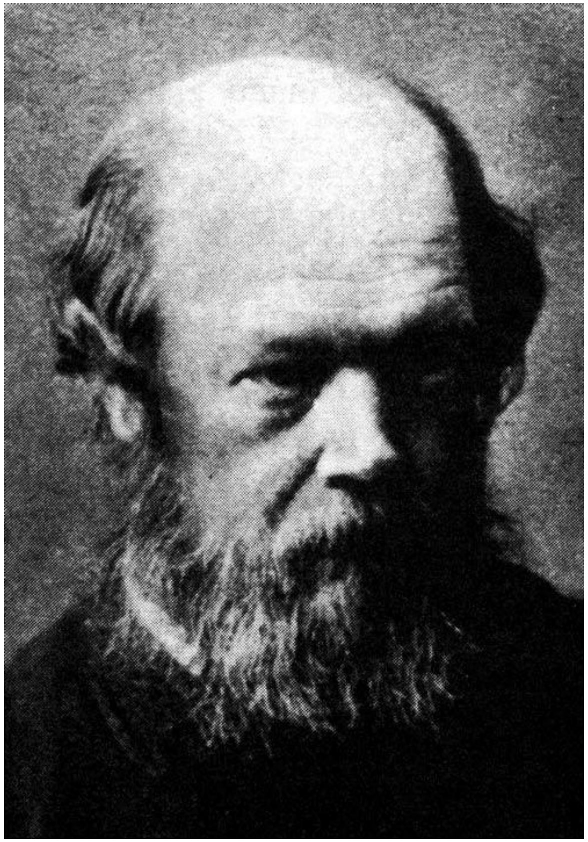


The “unknot”

(a mathematician’s “joke”)

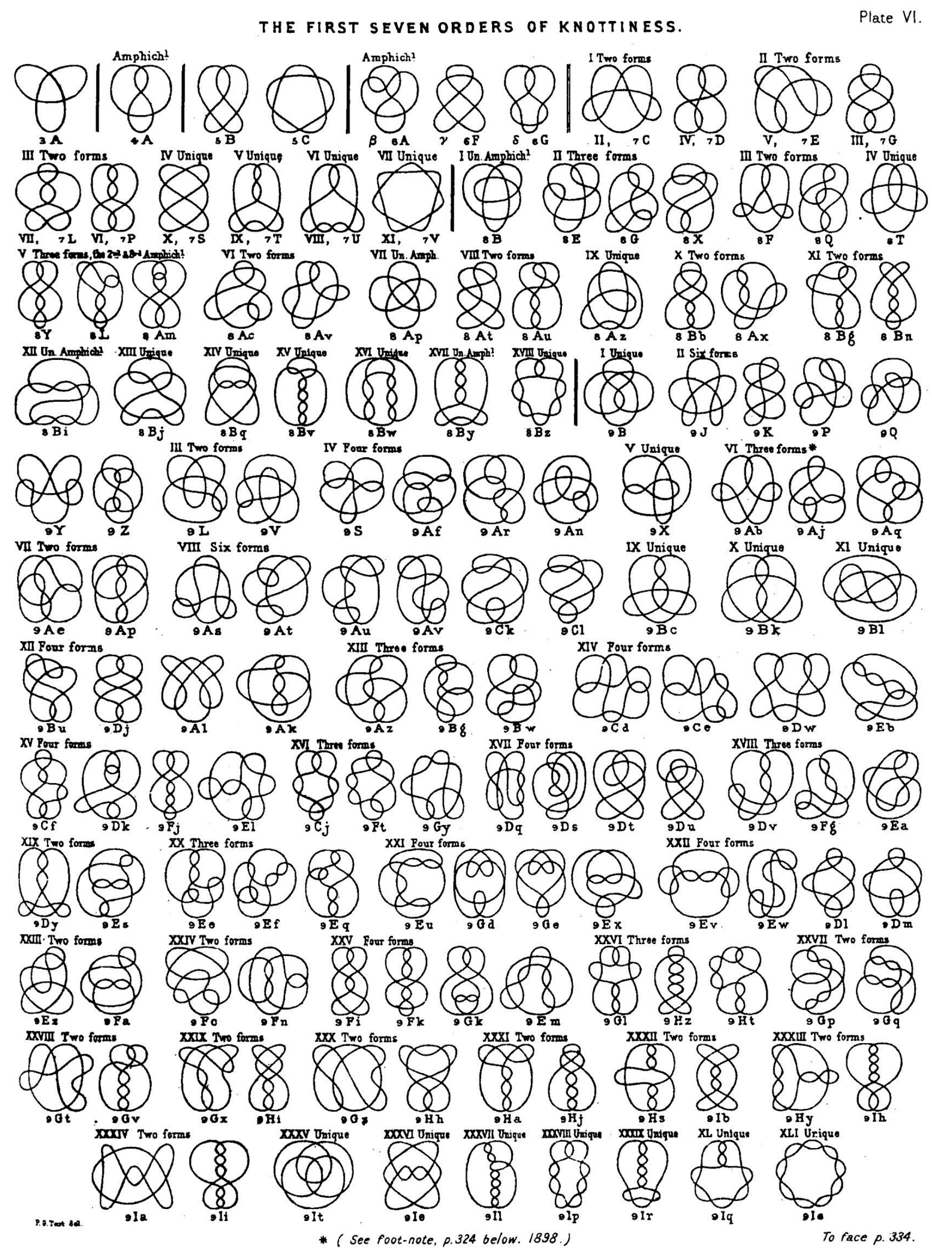


There are lots and lots of knots …

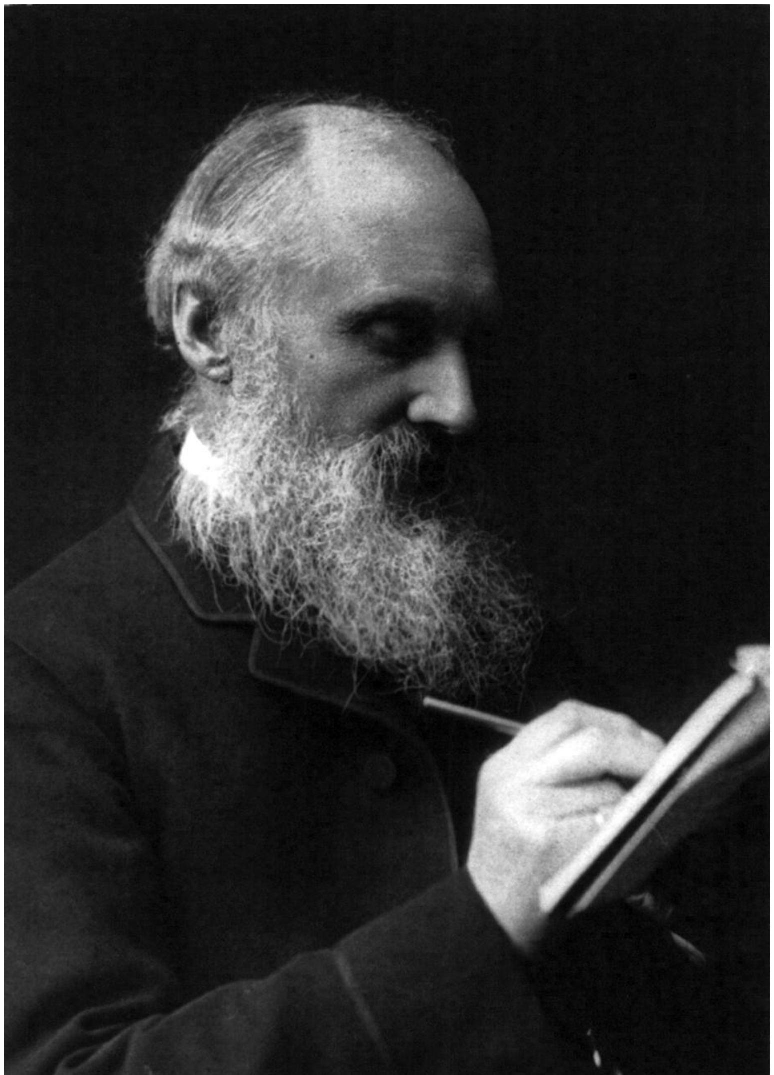


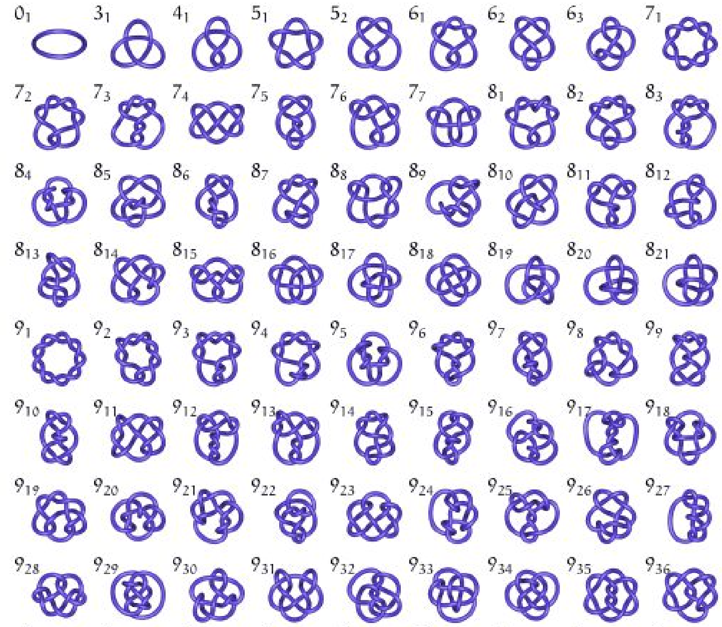
Peter Guthrie Tait

Tait’s dates: 1831-1901

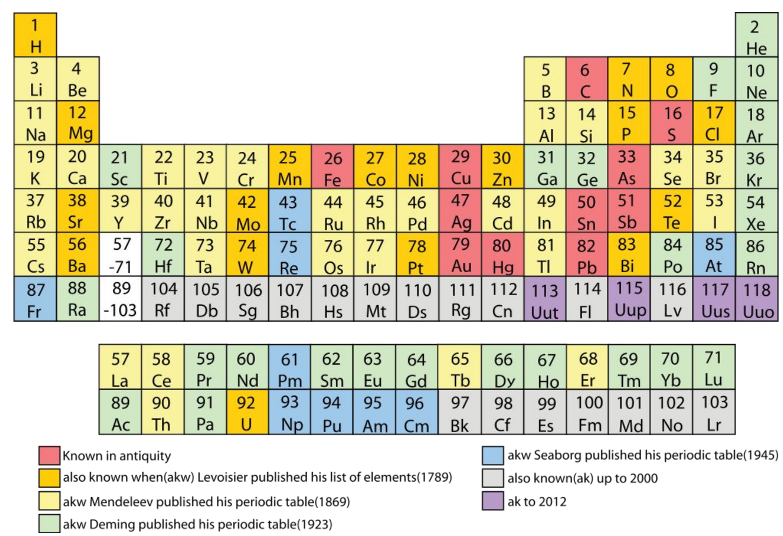


Lord Kelvin
(William Thomson)

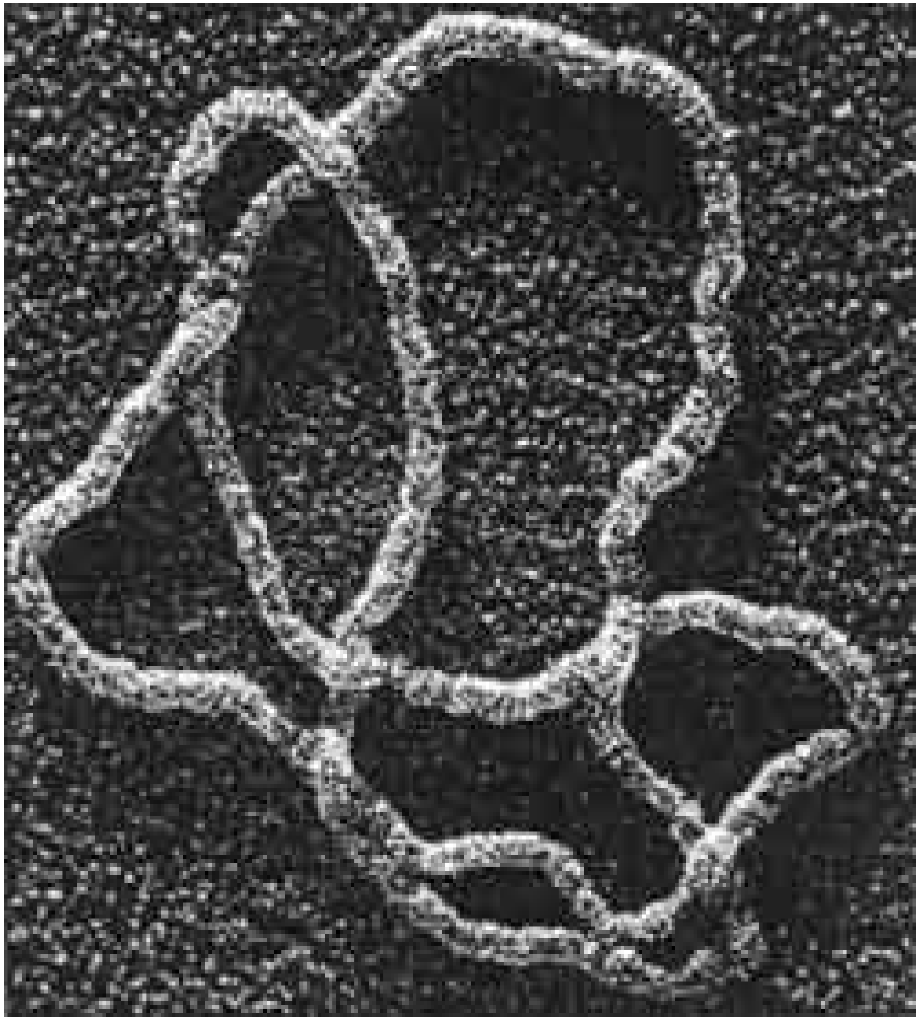




?

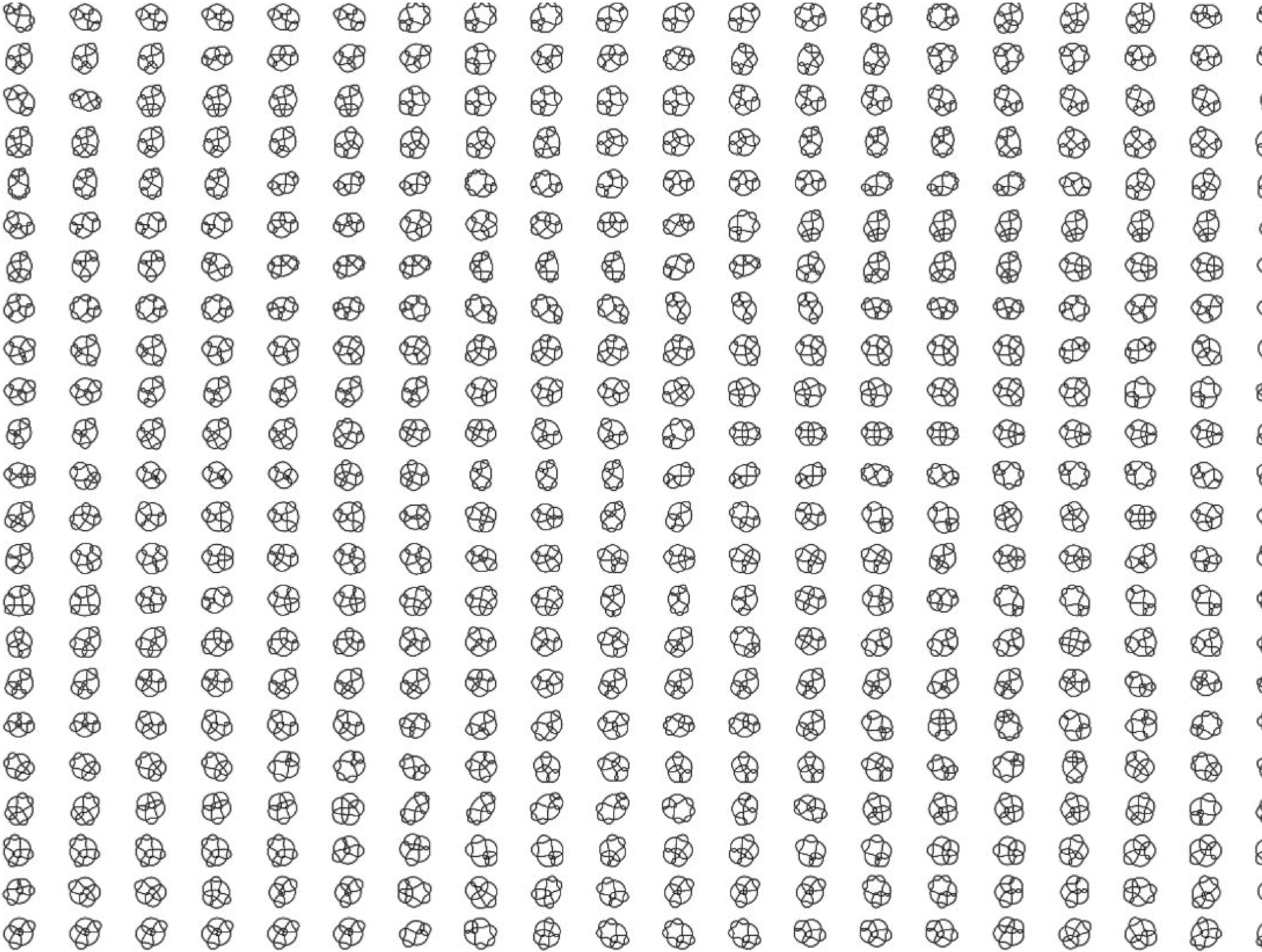


Knots don't explain the periodic table but….



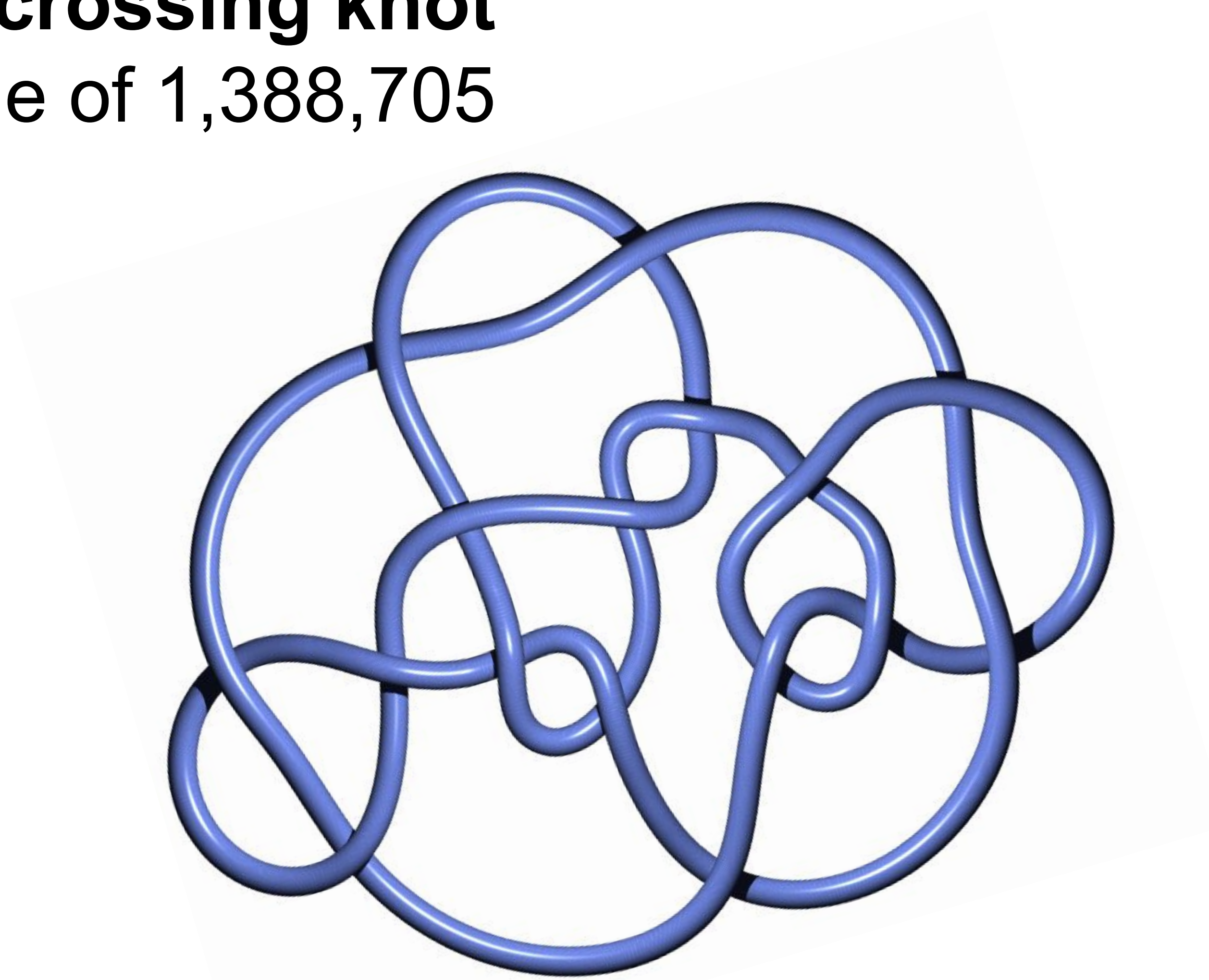
They do appear to be important in nature. Here is some knotted DNA

Science 229, 171 (1985); copyright AAAS



A 16-crossing knot

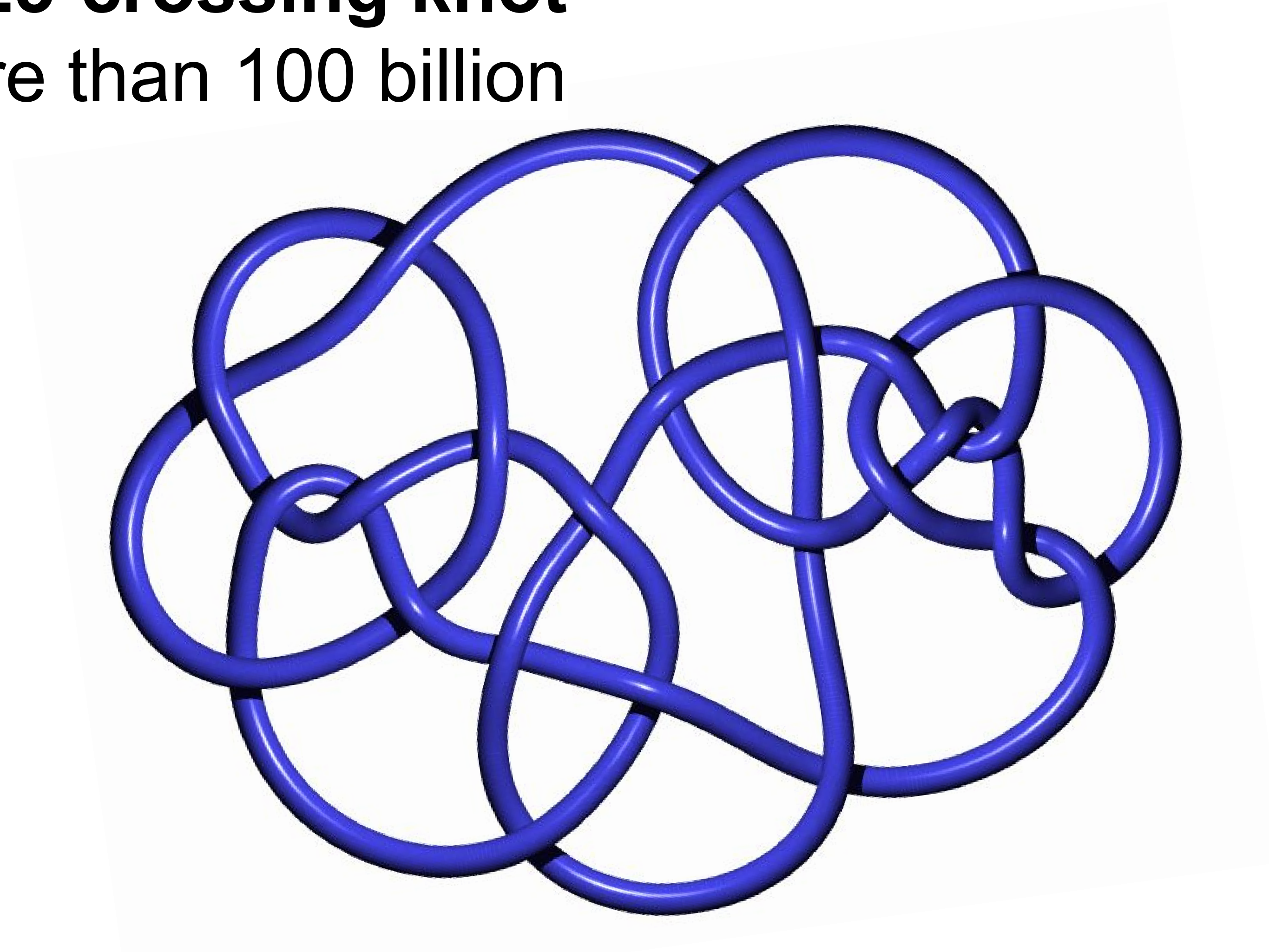
one of 1,388,705



The knot with archive number
16n-63441

Image generated at http:
//knotilus.math.uwo.ca/

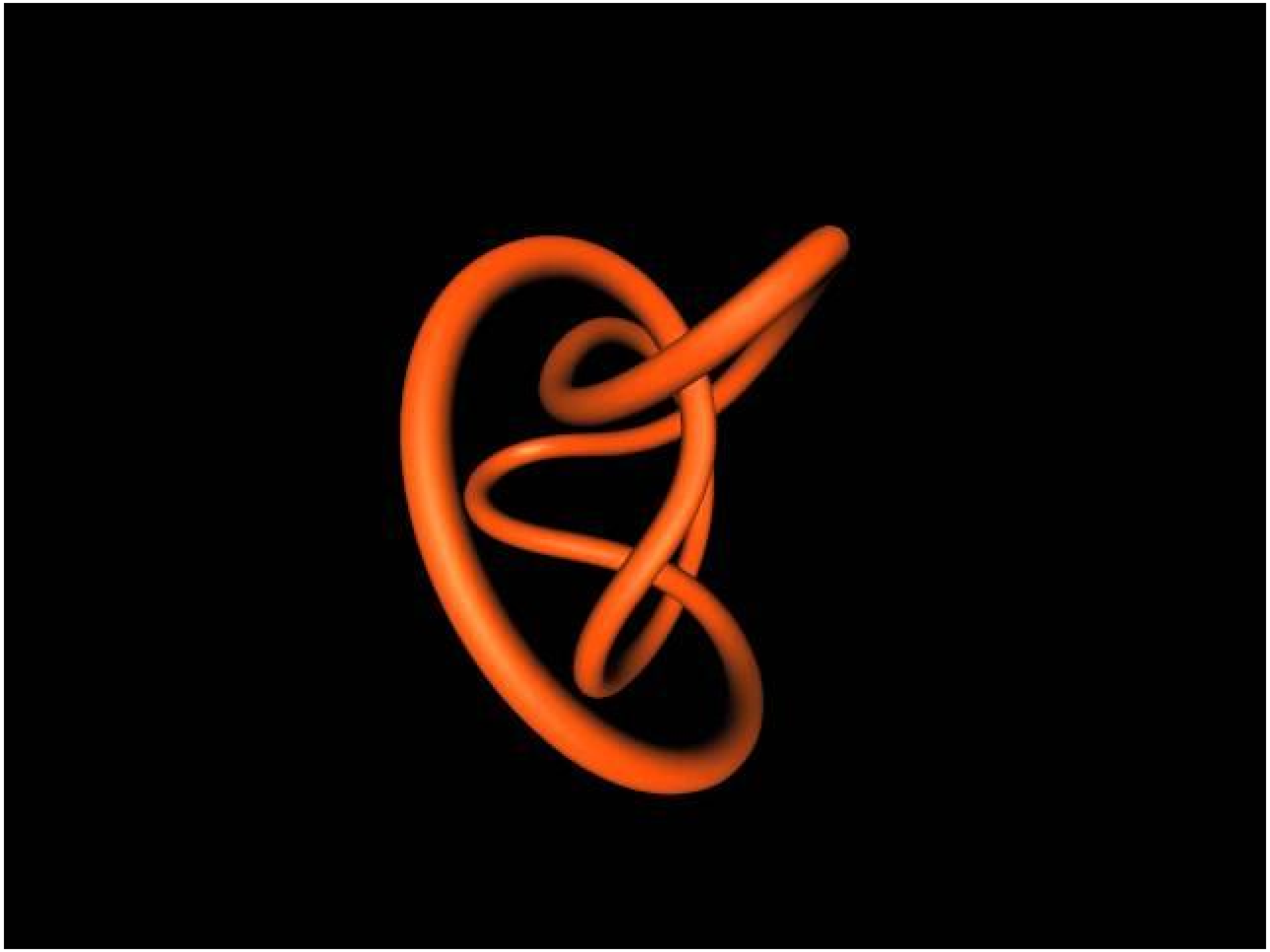
A 23-crossing knot
one of more than 100 billion



The knot with archive number
23x-1-25182457376

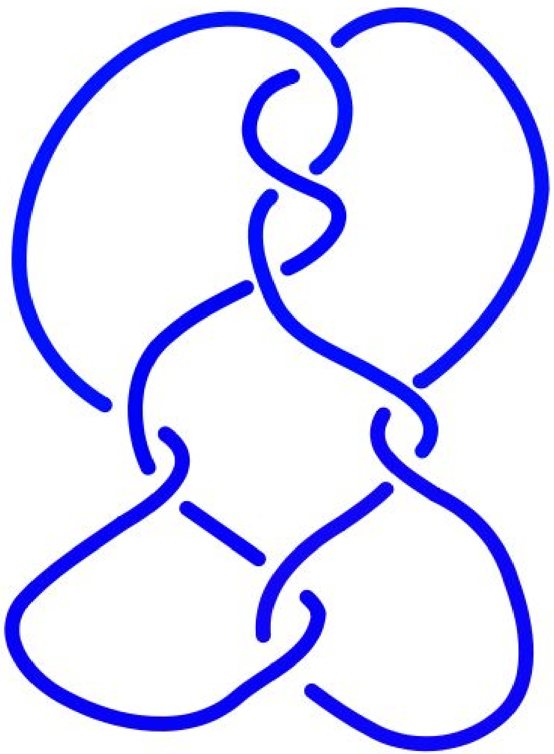
Image generated at http://knotilus.math.uwo.ca/

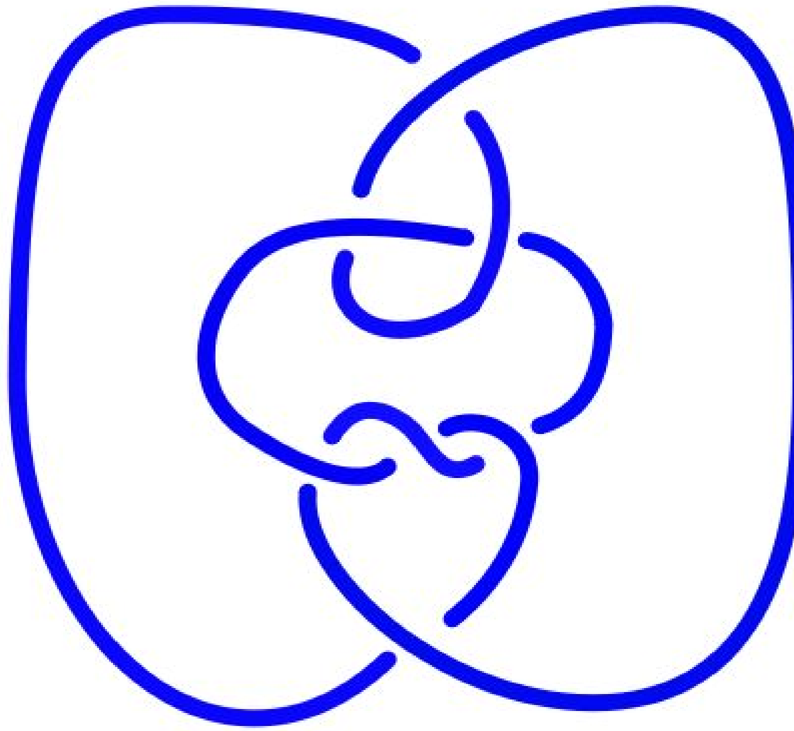
Spot the knot



Video: Robert Scharein
knotplot.com

Some other hard unknots





Measuring Topological Complexity

- We need certificates of topological complexity.
- Things we can compute from a particular instance of (in this case) a knot, but that does change under deformations.
- You already should know one example of this.
    - The linking number from E&M.

What kind of tools do we need?

1. Methods for encoding knots (and links) as well as rules for understanding when two different codings are give equivalent knots.
2. Methods for measuring topological complexity. Things we can compute from a particular encoding of the knot but that don’t agree for different encodings of the same knot.

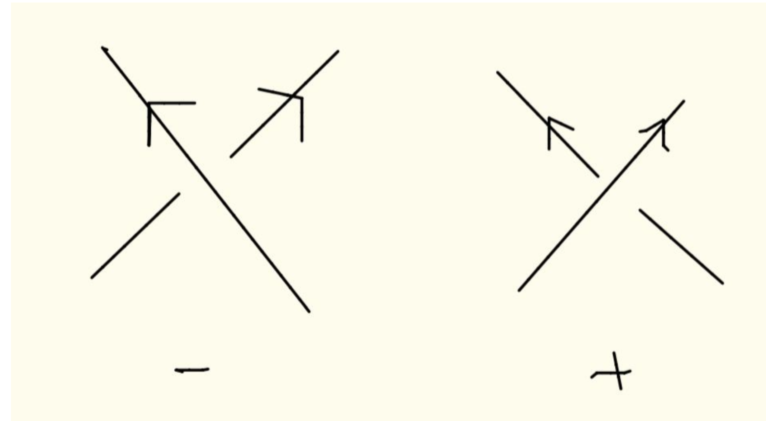
Knot Projections

A typical way of encoding a knot is via a **projection**.

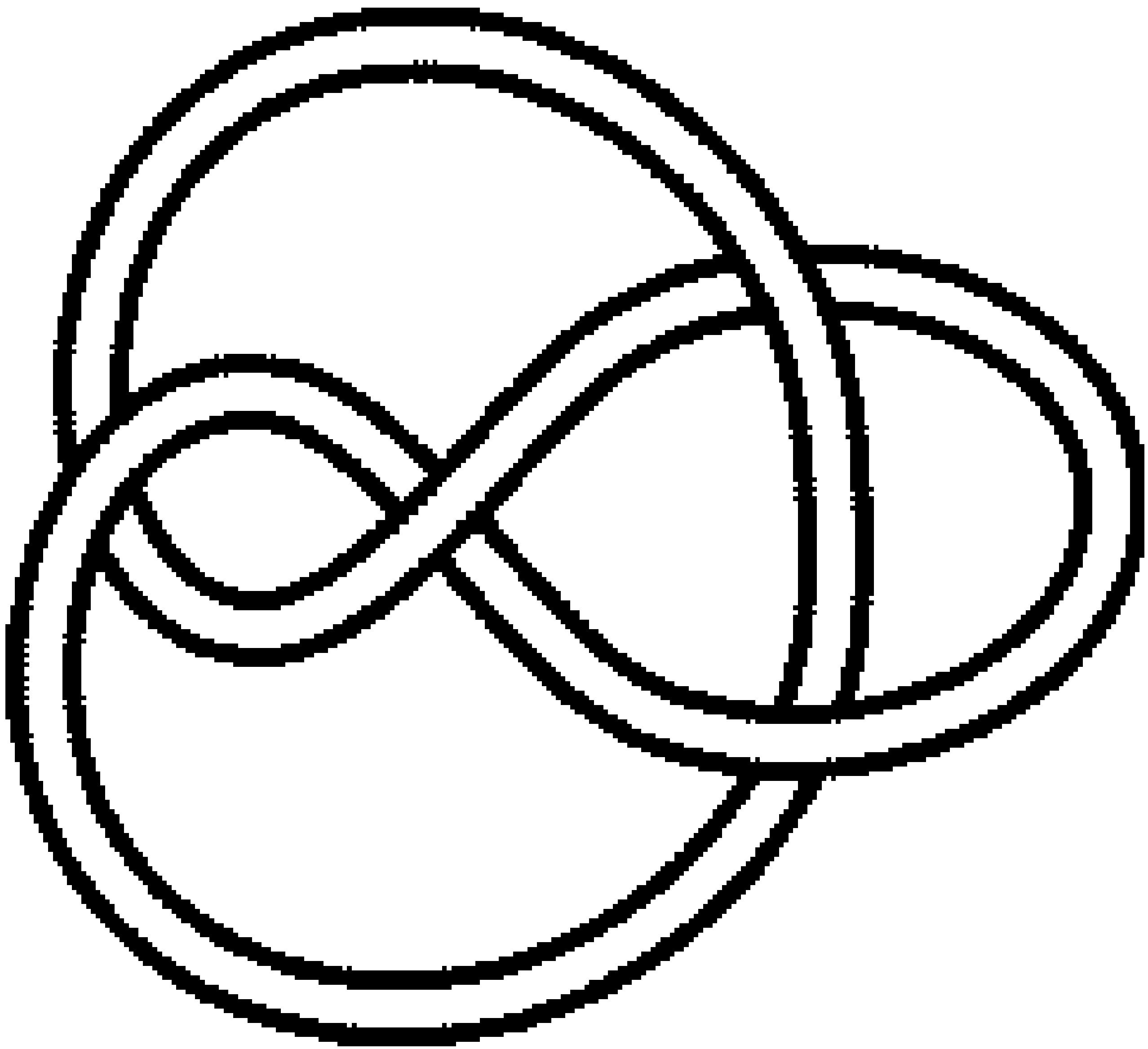
We imagine the knot **K** sitting in 3-space with coordinates (x,y,z) and project to the xy-plane. We remember the image of the projection together with the over and under crossing information as in some of the pictures we just saw.

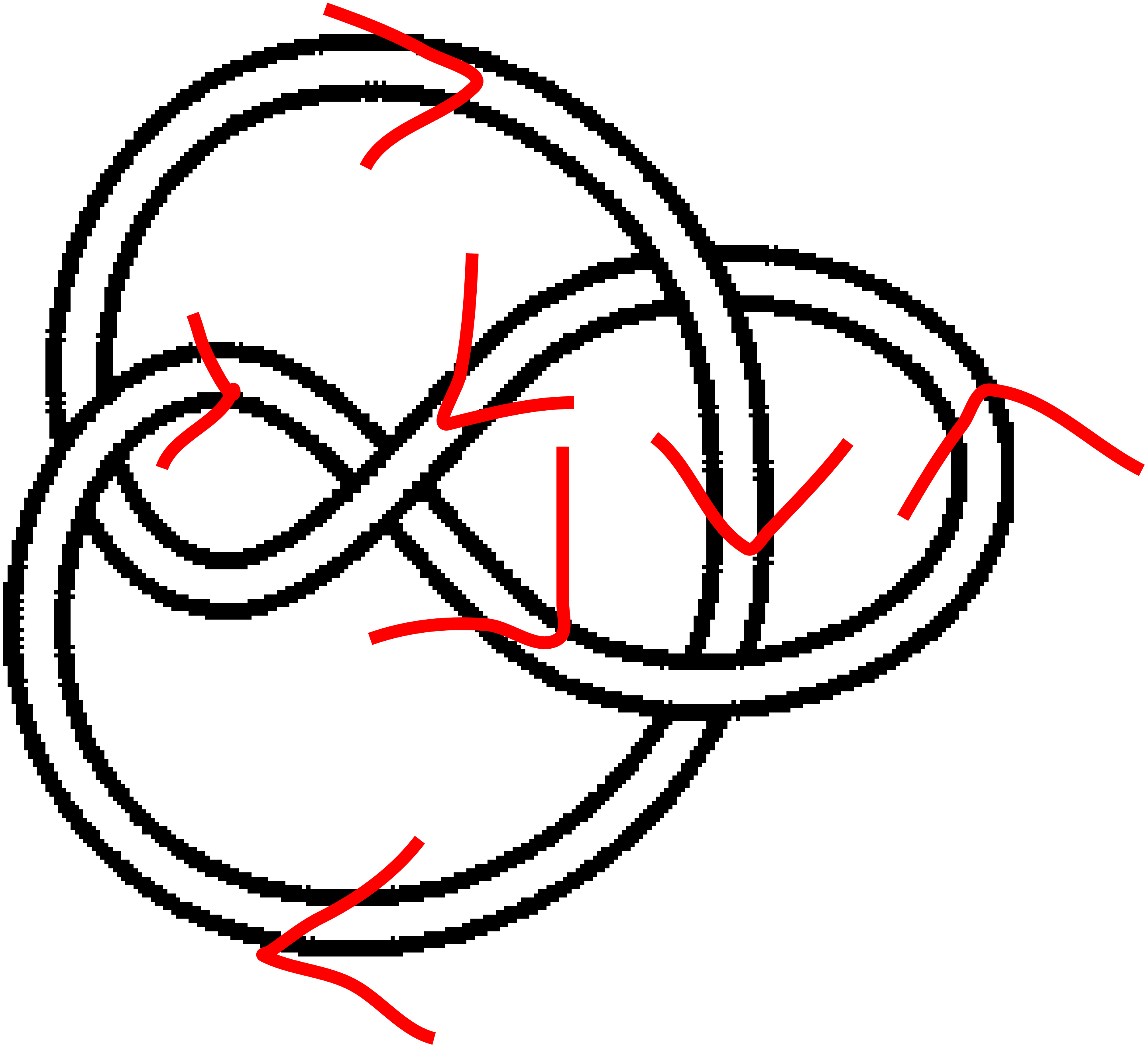
Some basic properties of knot projections

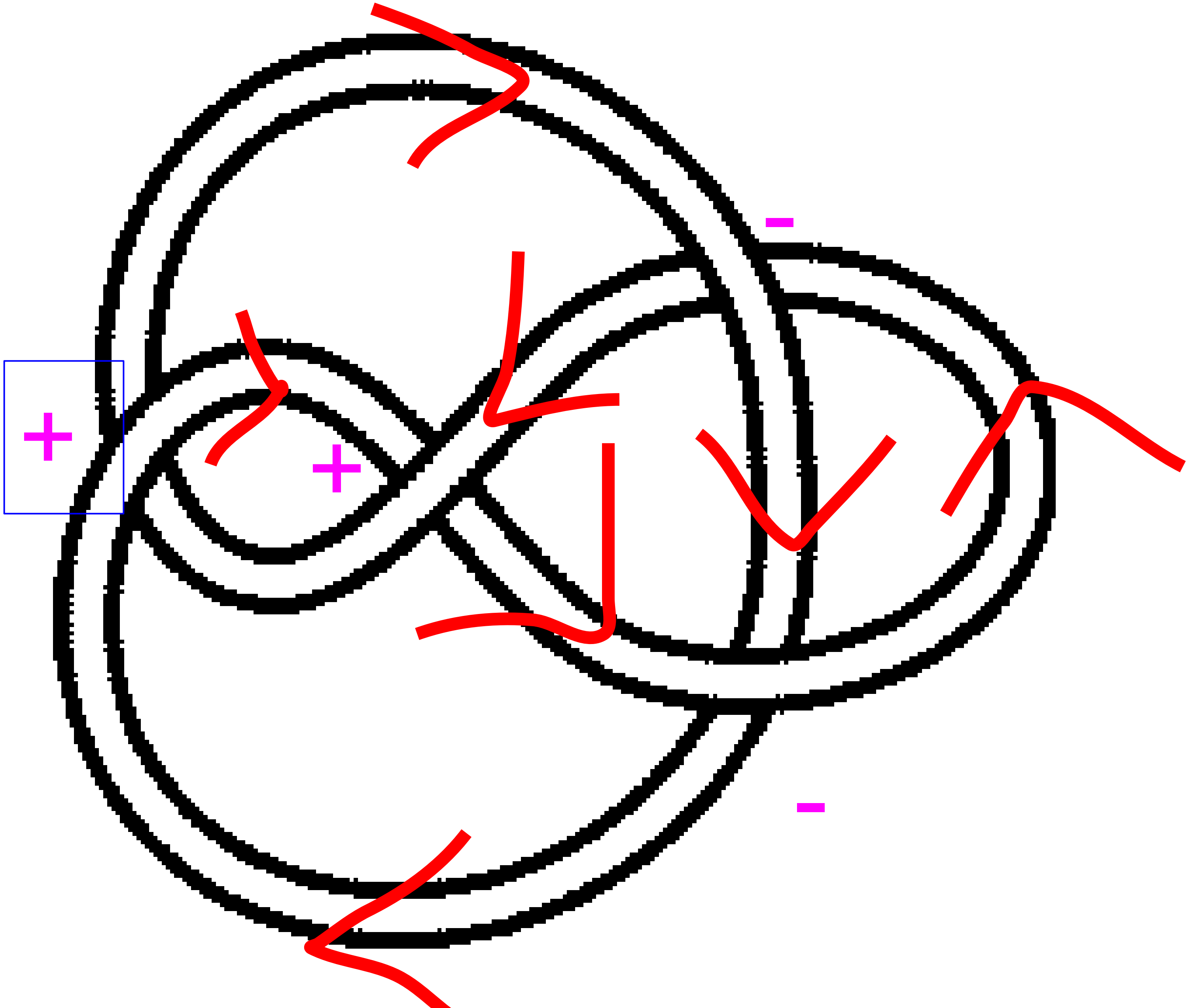
- Sign of a crossing:

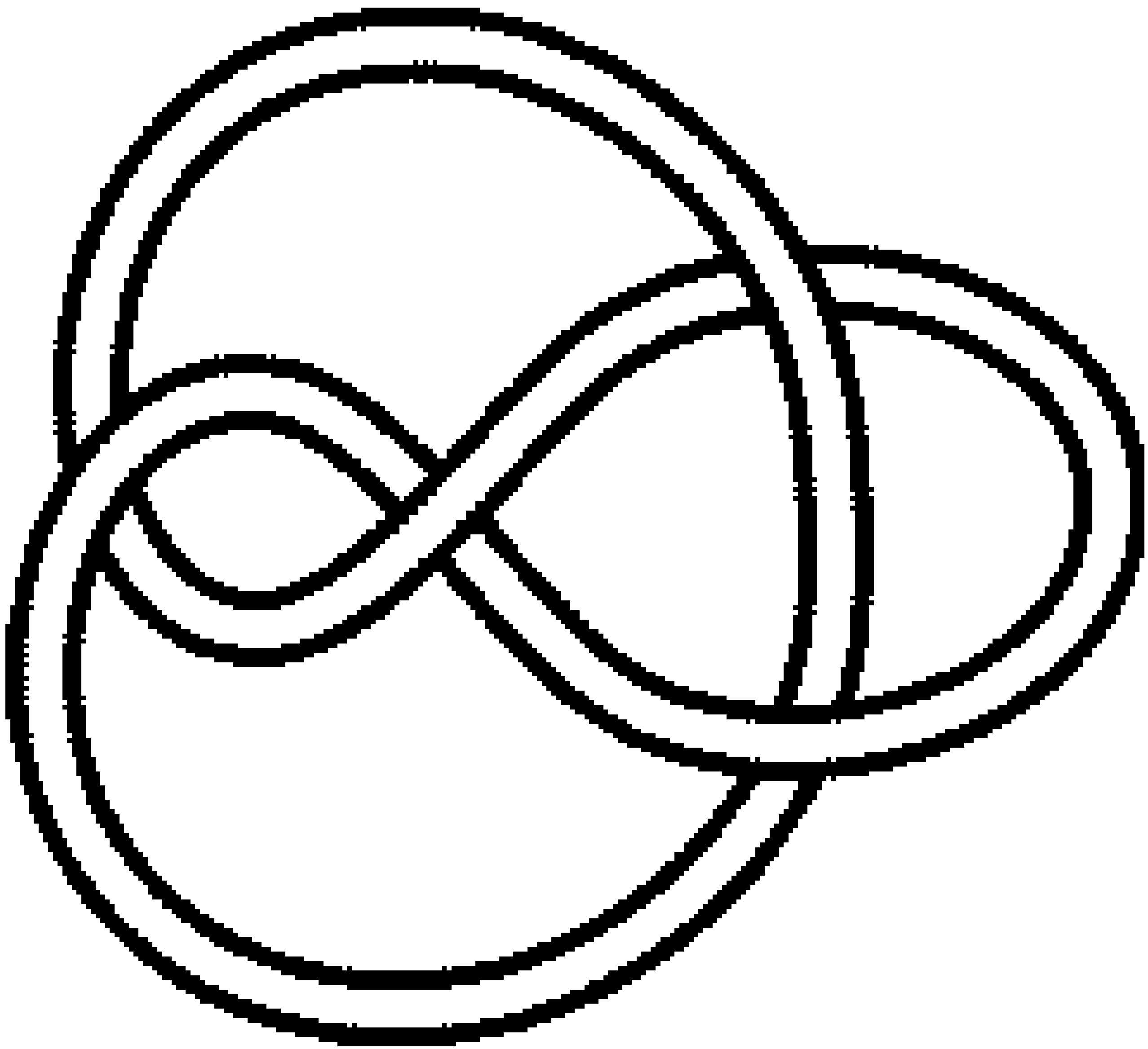


Projections can be alternating: as you move along the knot the over and under crossings alternate

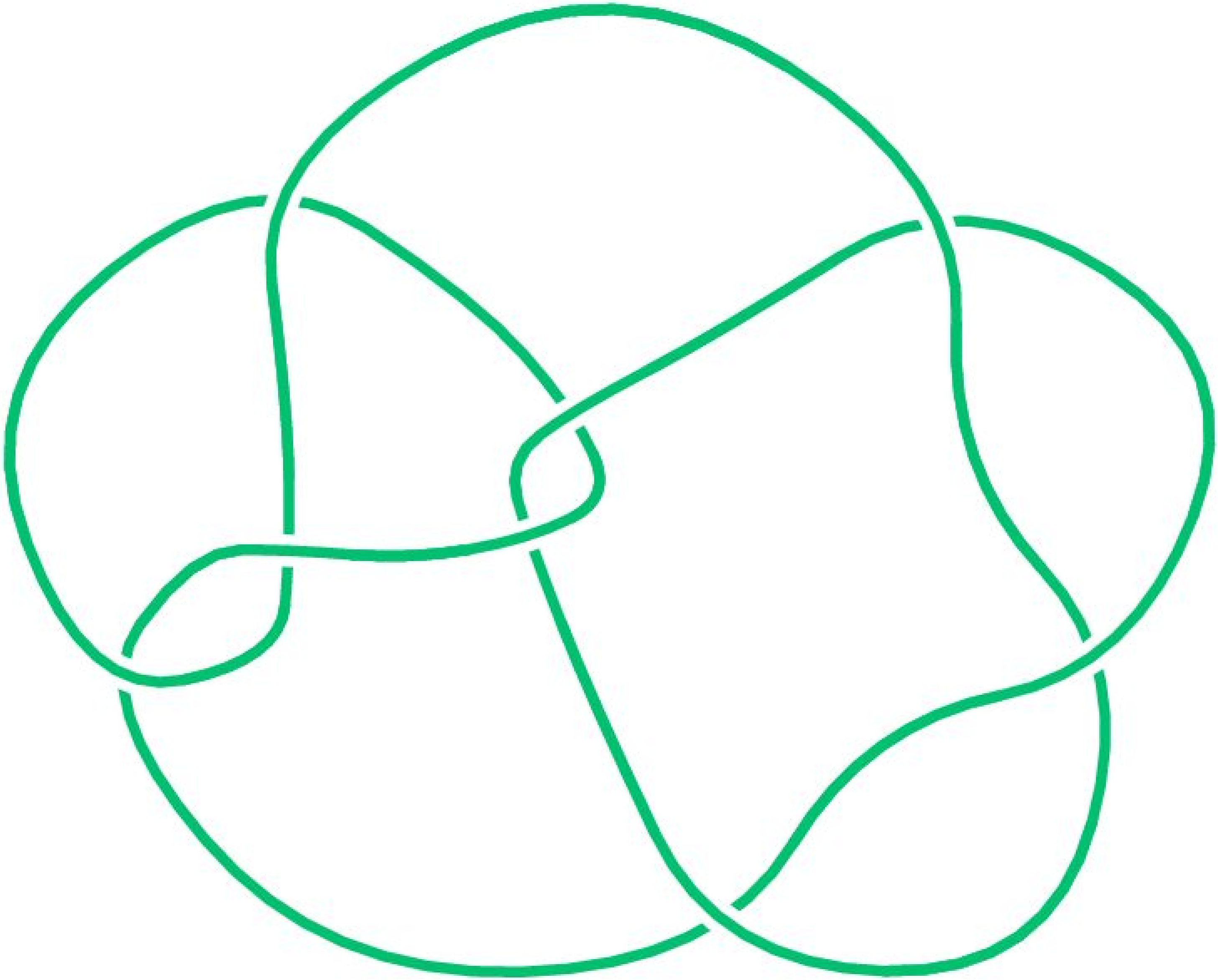


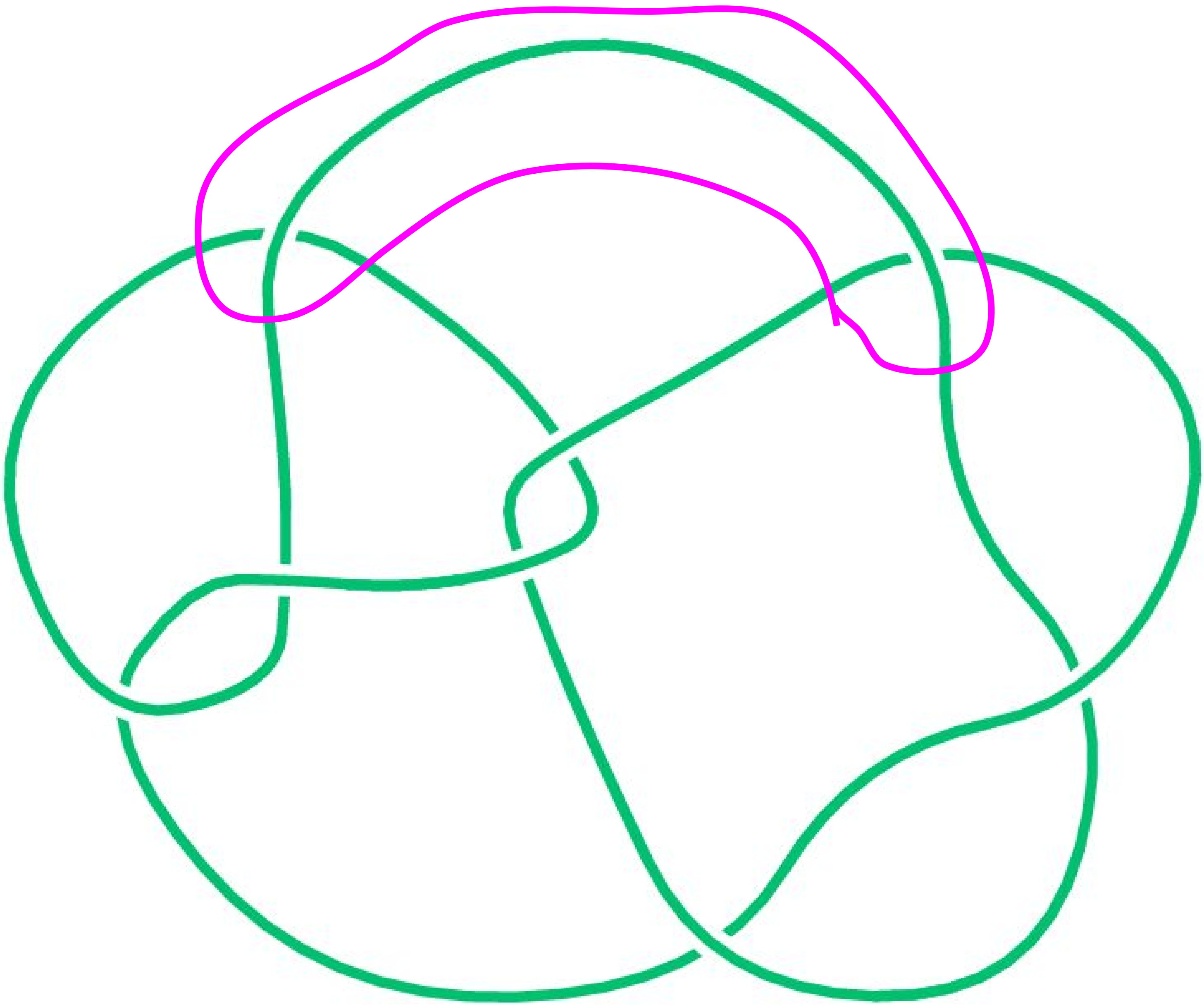






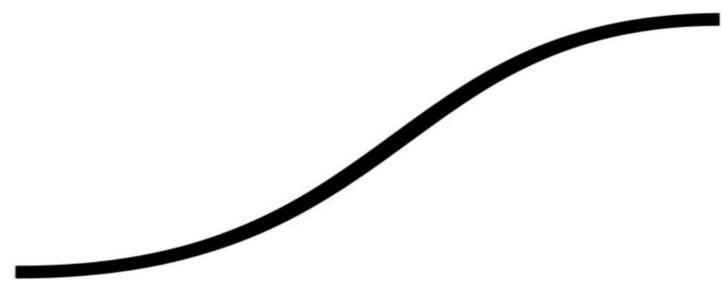
This is an alternating knot.



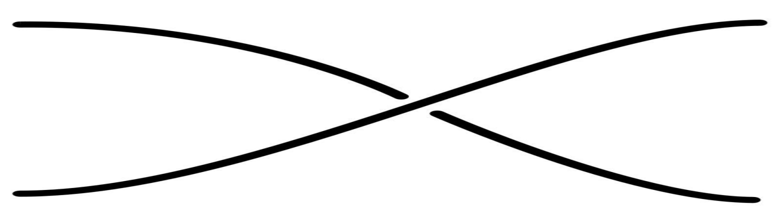


Not alternating!

Any “generic” picture of a knot only has two picture near a given point on the knot. Either a nice smooth arc.

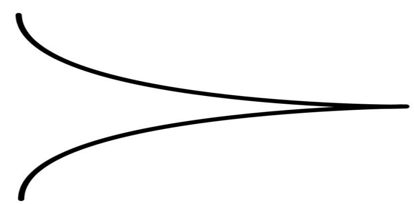


Or two smooth curve crossing over each each other with distinct tangent lines.

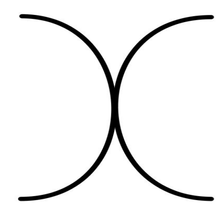


In particular we don't have:

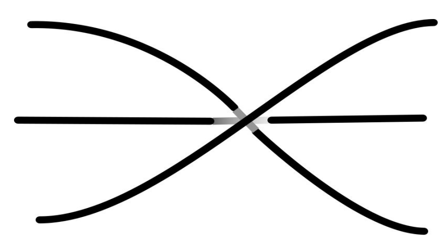
Cusps (where the knot has vertical tangent.)



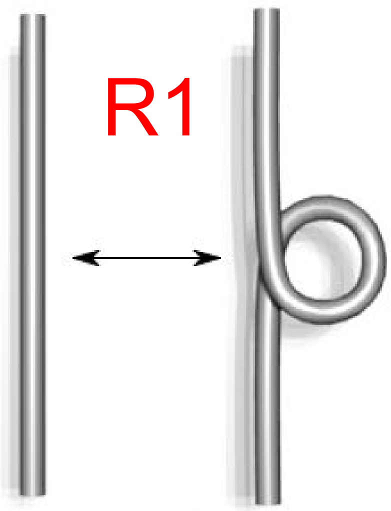
Common tangents.

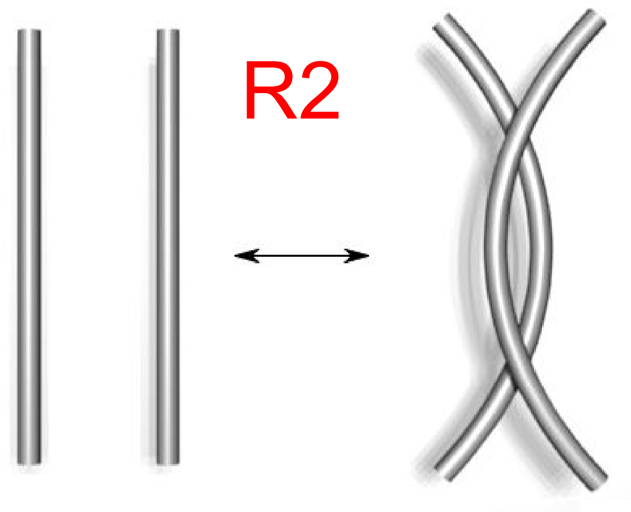


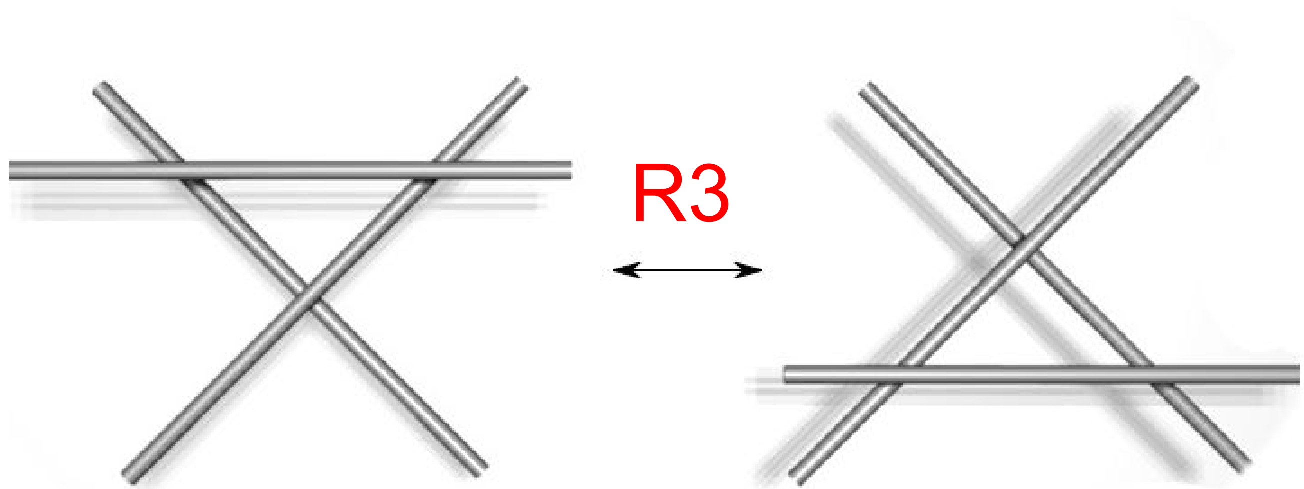
Or triple points.



Reidermeister Moves:



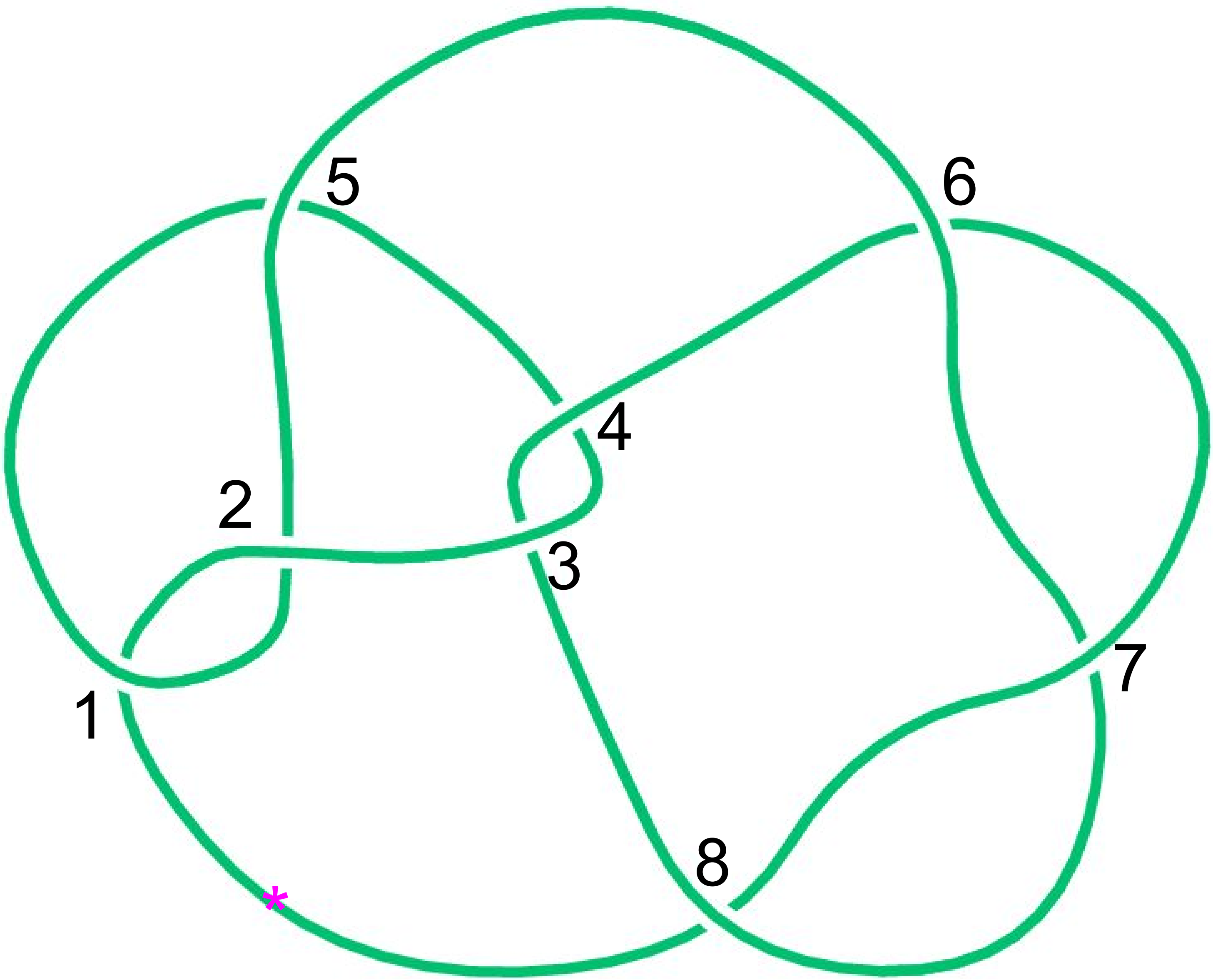




Other ways of specifying a knot

Gauss Code or Gauss Diagram: Label the crossings (say by 1,2,...n).

1. Pick a base point on the knot and an orientation.
2. Run along the knot and write down the label of the crossing you meeting together with a O/U if you are going over or under.

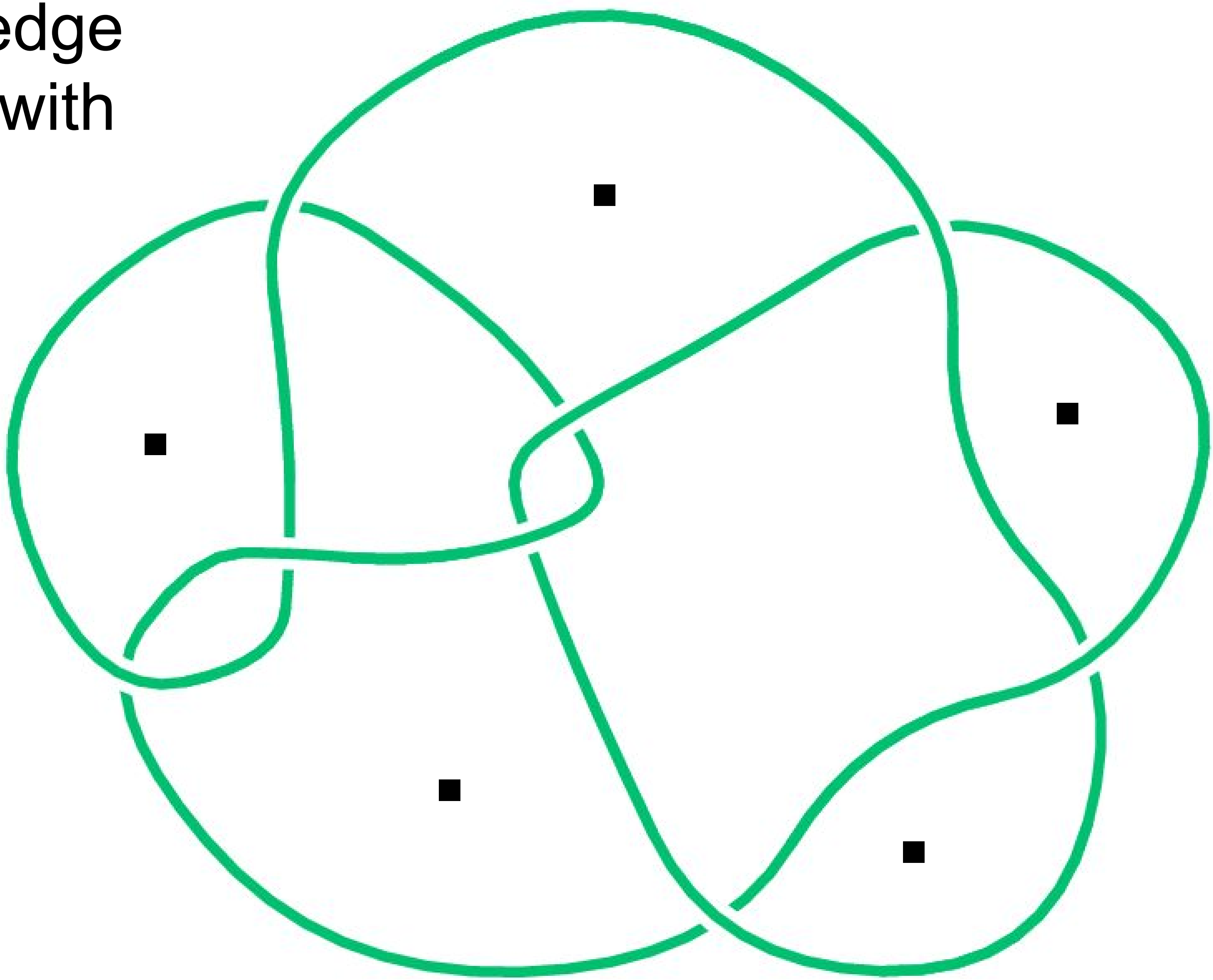


1u2o3o4u5u1o2u5o6o7u8o3u4o6u7o8u

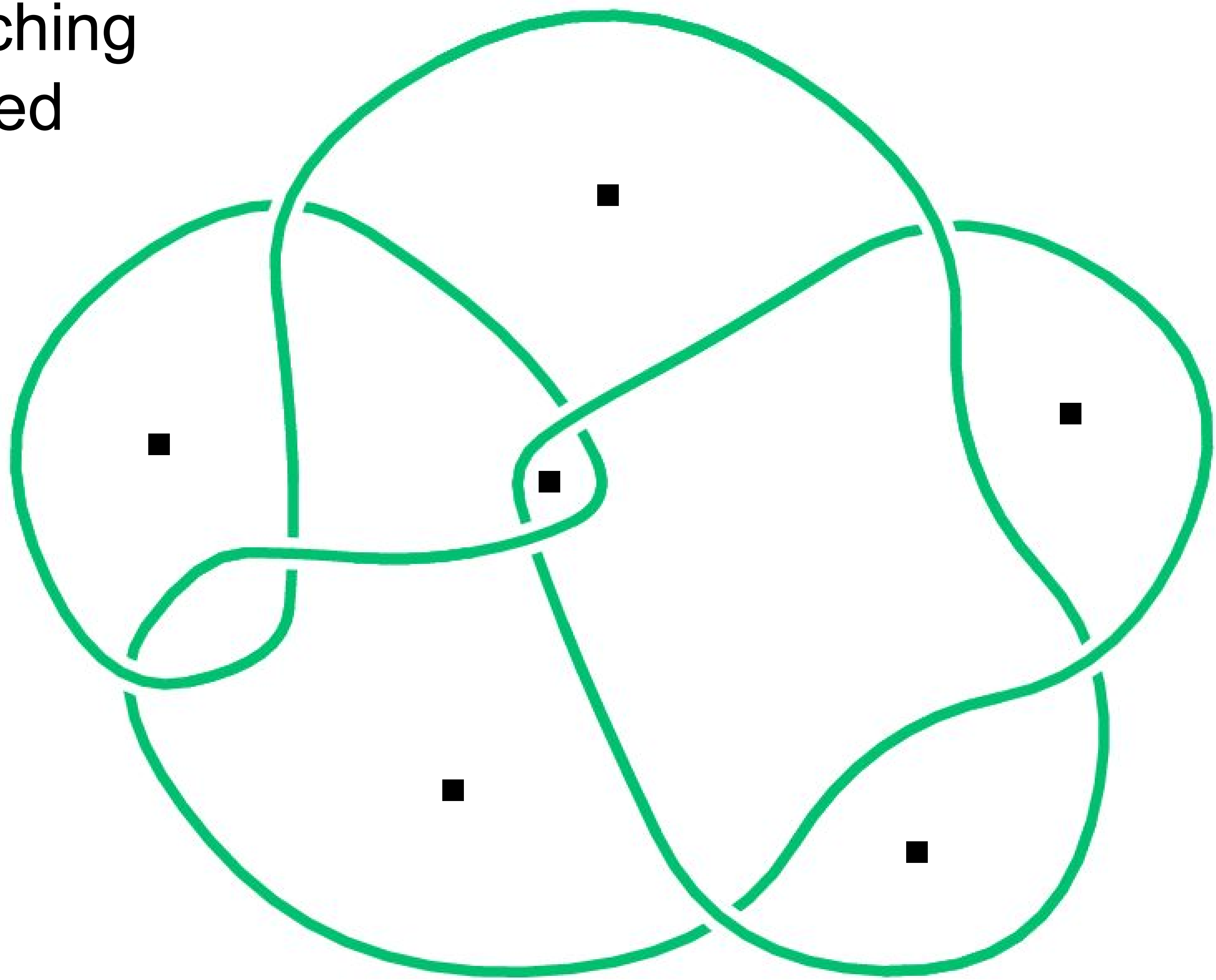
Signed planar graphs:

Black and white color the knot complement.

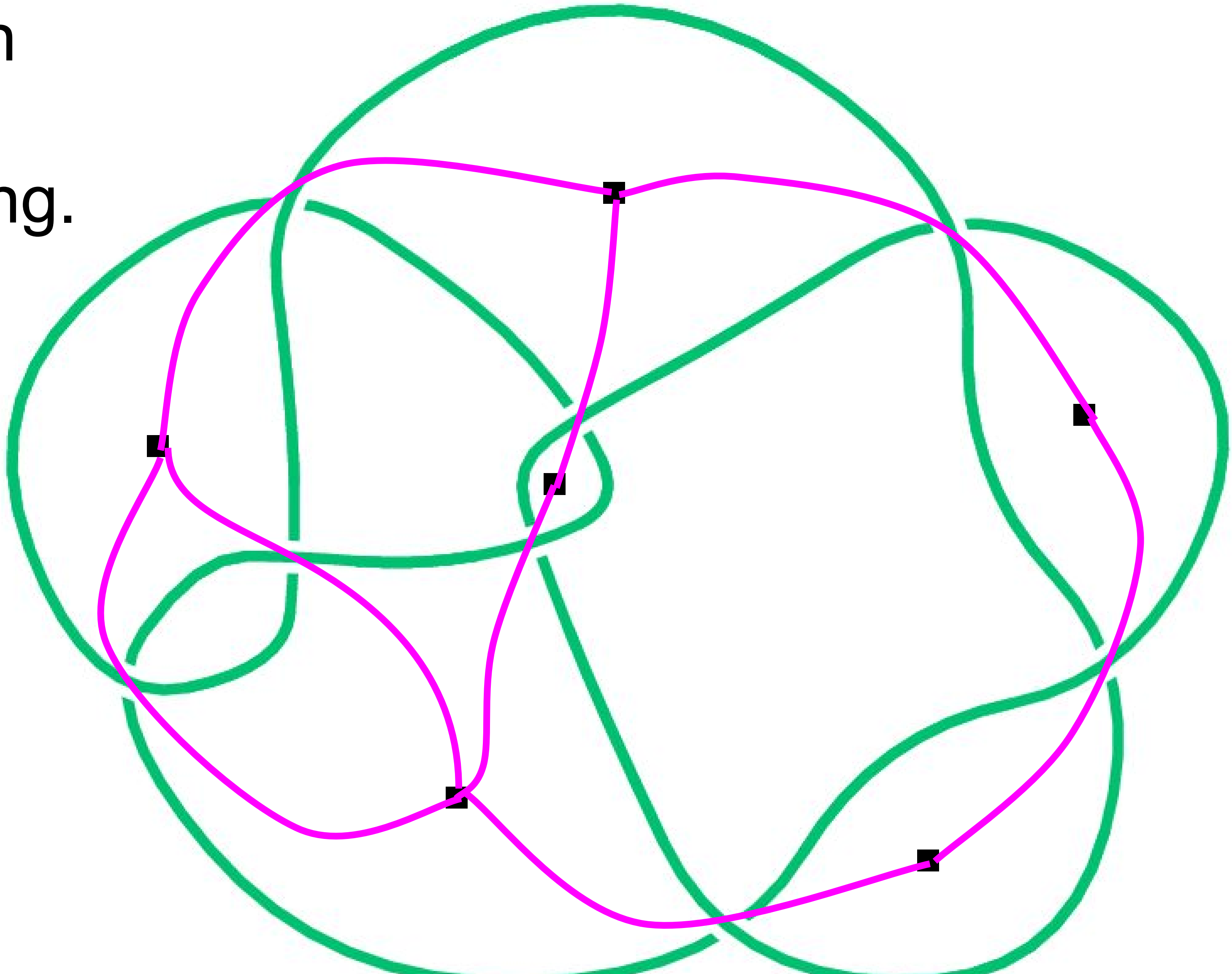
Label all regions sharing an edge the exterior with a black dot.



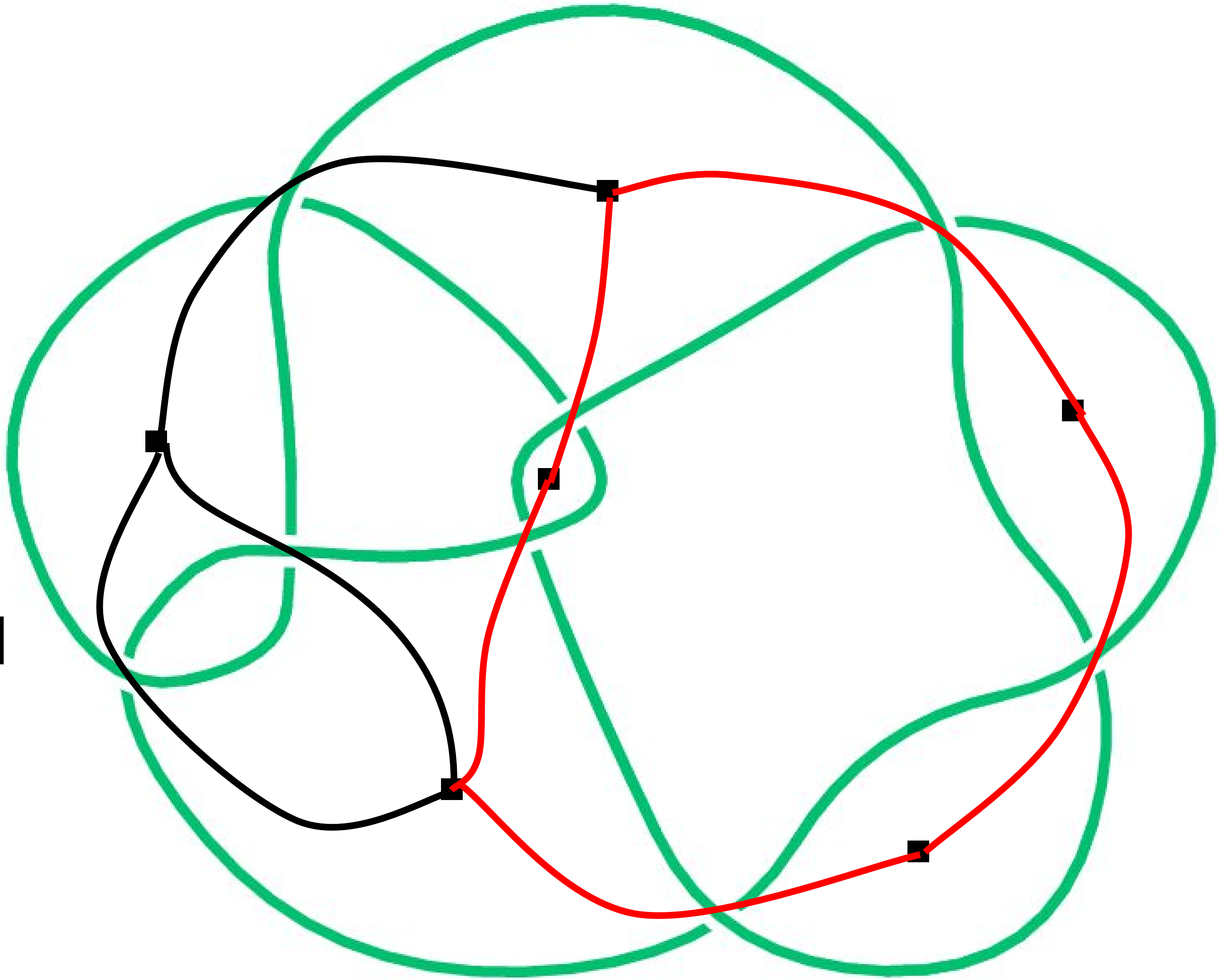
Now label all regions touching a black dotted region at a crossing with a black dot. Keep going as required.



Join all black dots through the
each crossing.



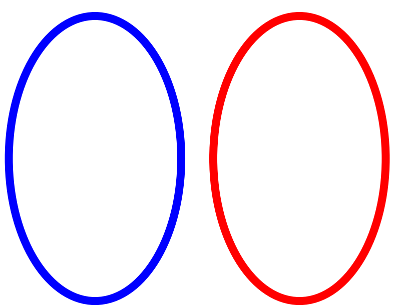
Label the edges with a + if the twist going from black dotted regions is right-handed and - if left-handed. (Here red is +, black -)



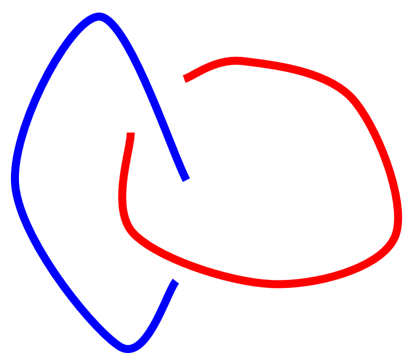
Topological Invariants:
The linking number.

Discovered by Gauss in studying Electricity and Magnetism.

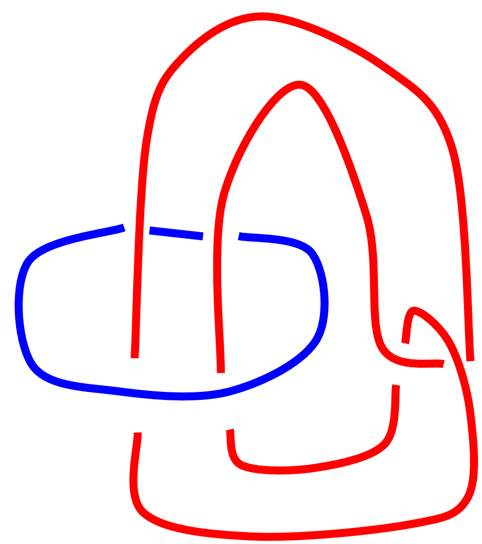
Can distinguish the unlink and the Hopf Link.



Unlink

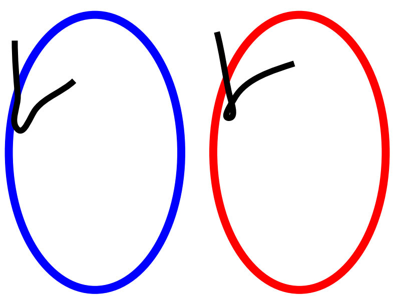


Hopf Link

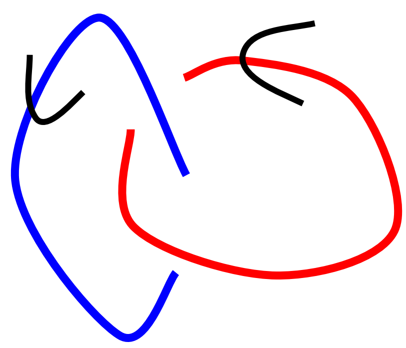


Whitehead Link?

To compute, orient both components.
Count the crossings of blue **over** red
with the signs as before.



Unlink



Hopf Link

Whitehead Link?

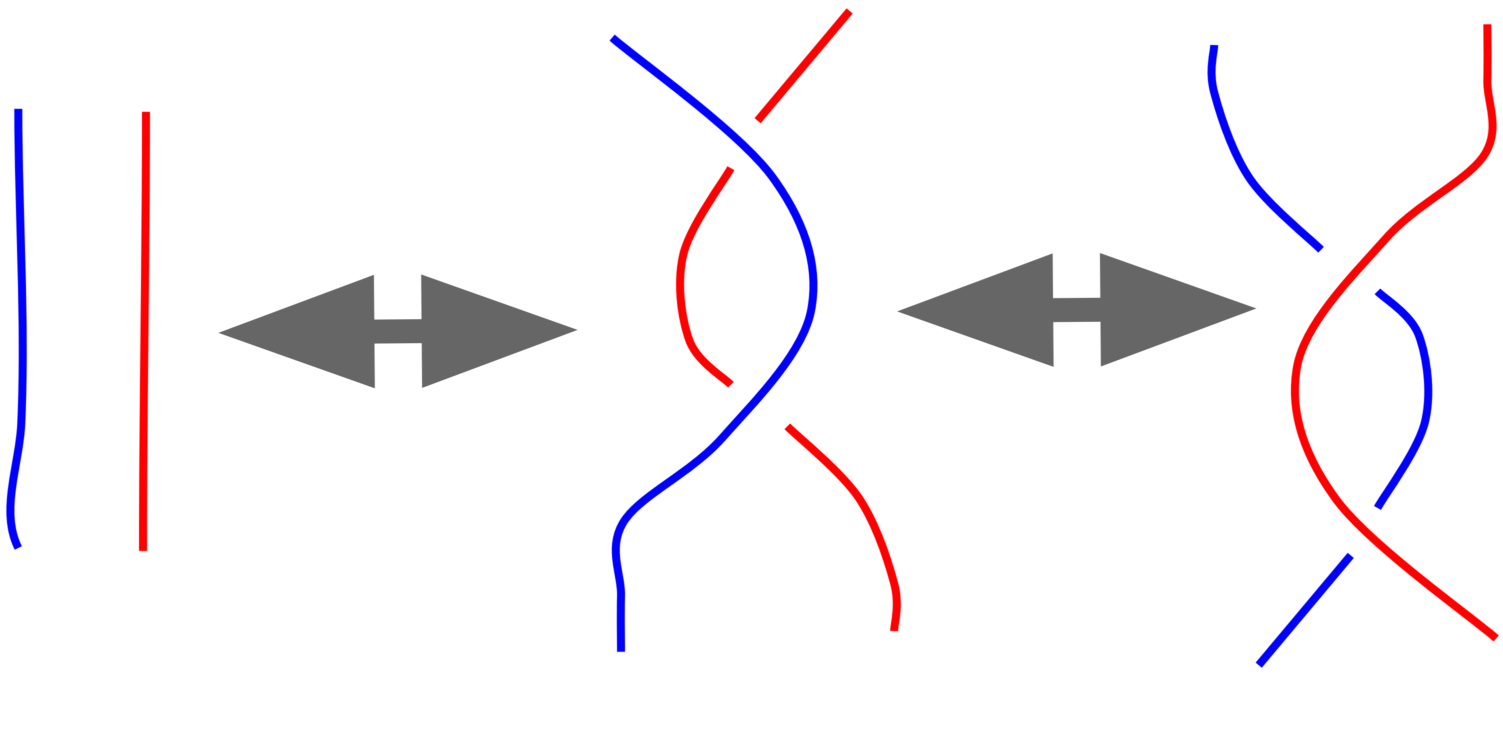
$LK(K_1,K_2)=1-1=0$

$LK(K_1,K_2)=+1$

$LK(K_1,K_2)=1-$

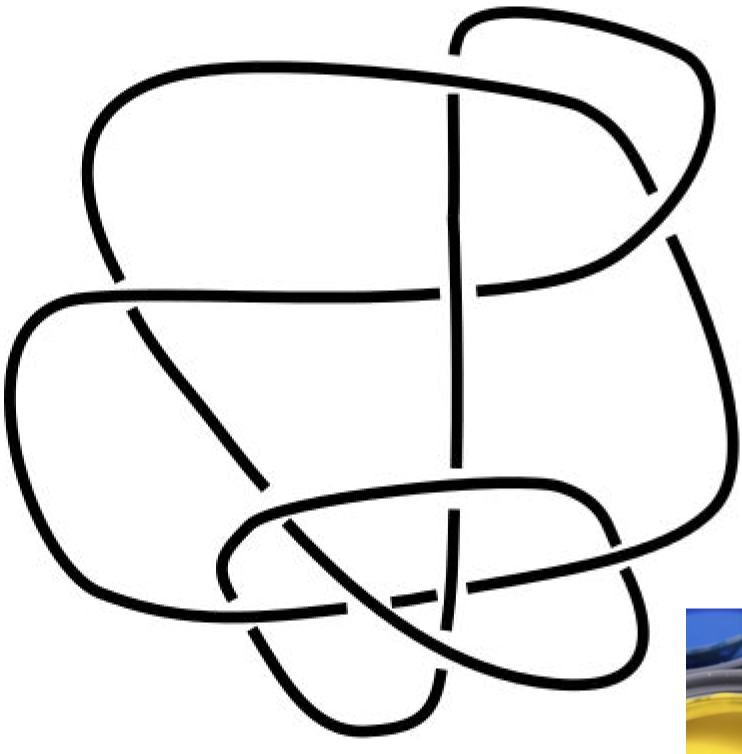
Why an invariant?

The only way the picture can change is:

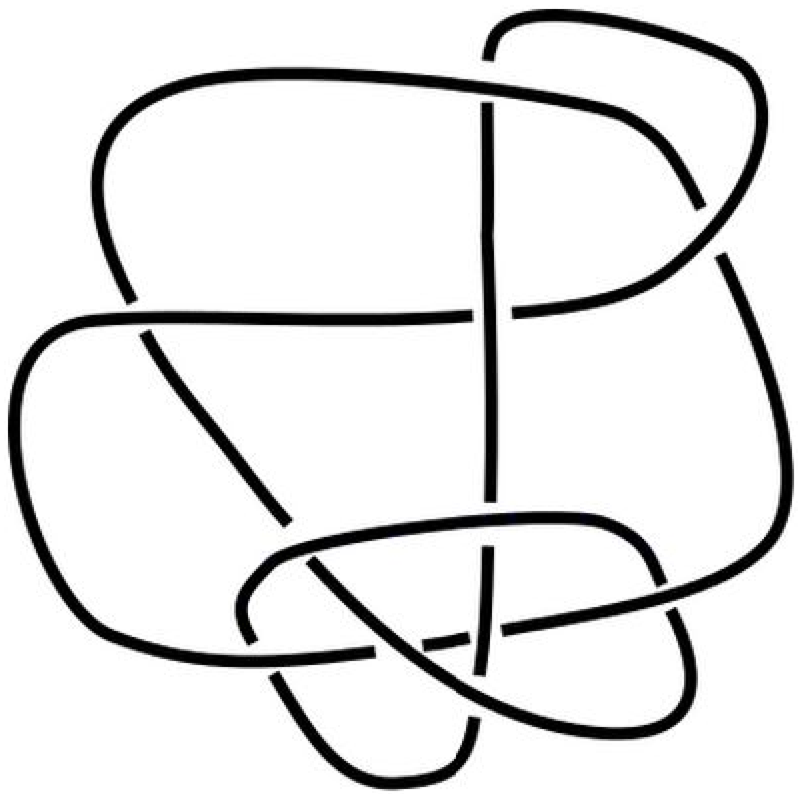


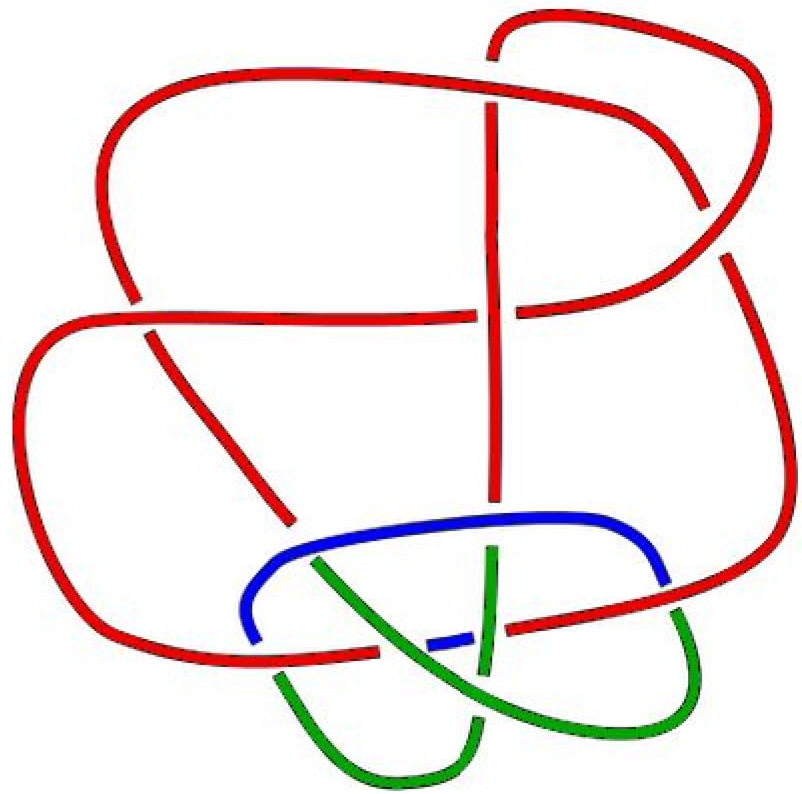
Both leave LK unchanged.

How do we know it is knotted?

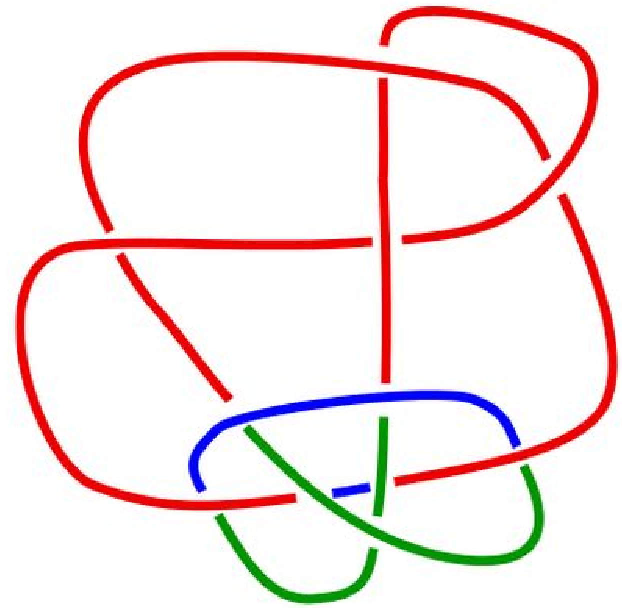


One way is by coloring it …

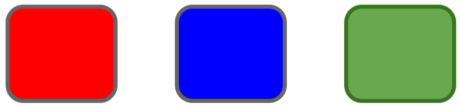




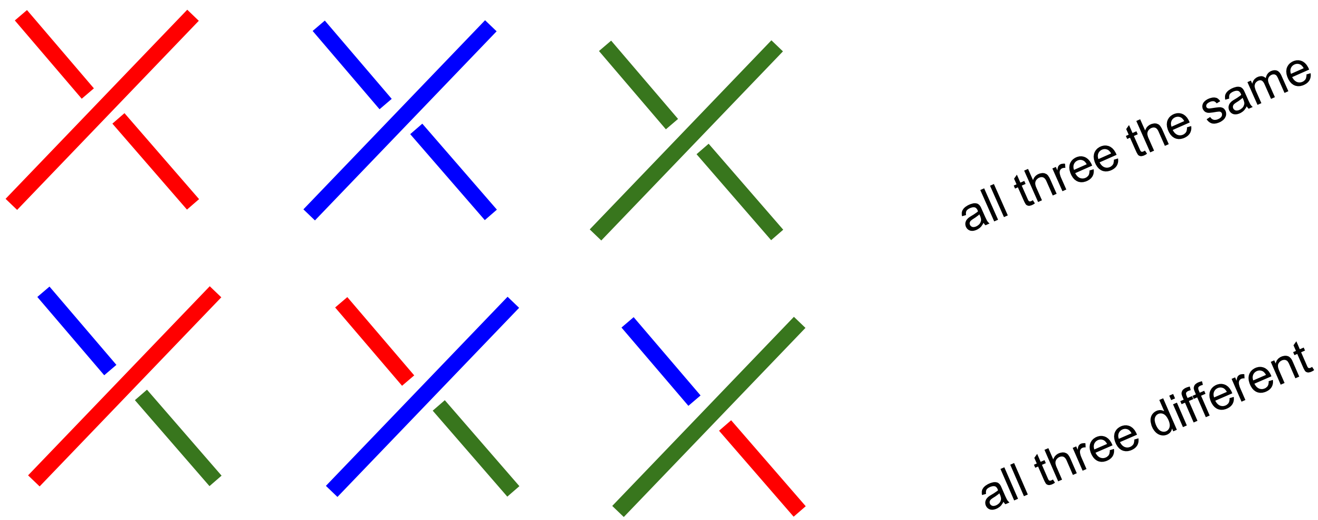
Rules of the coloring game:

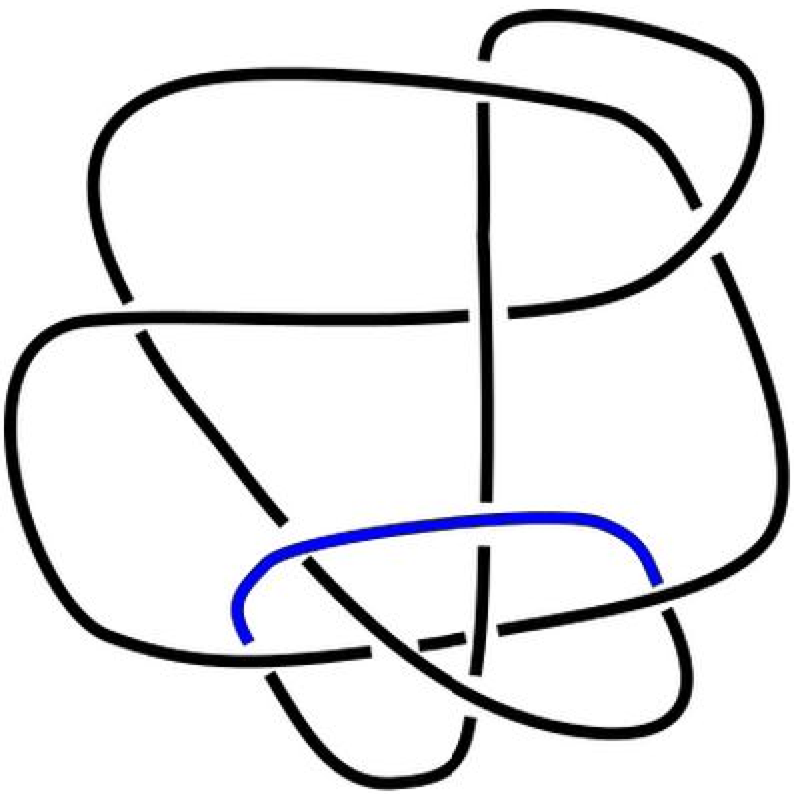


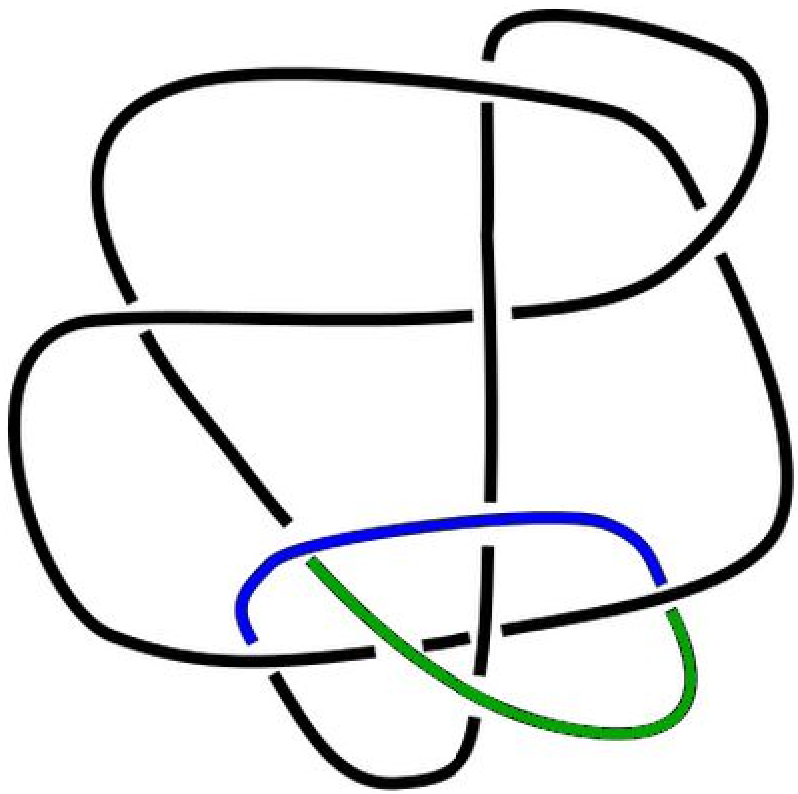
- only three colors (for now)

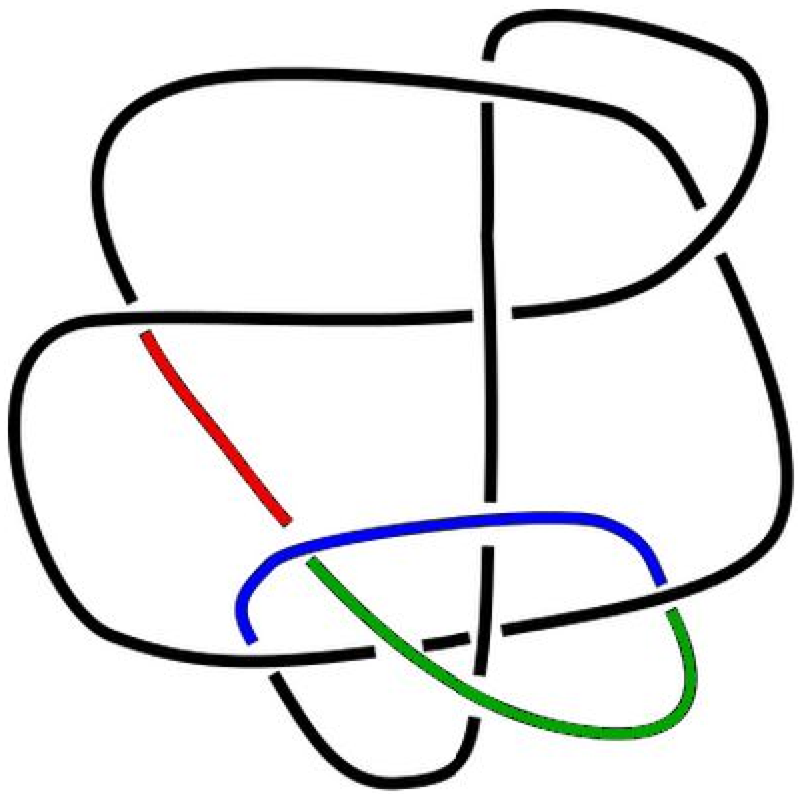


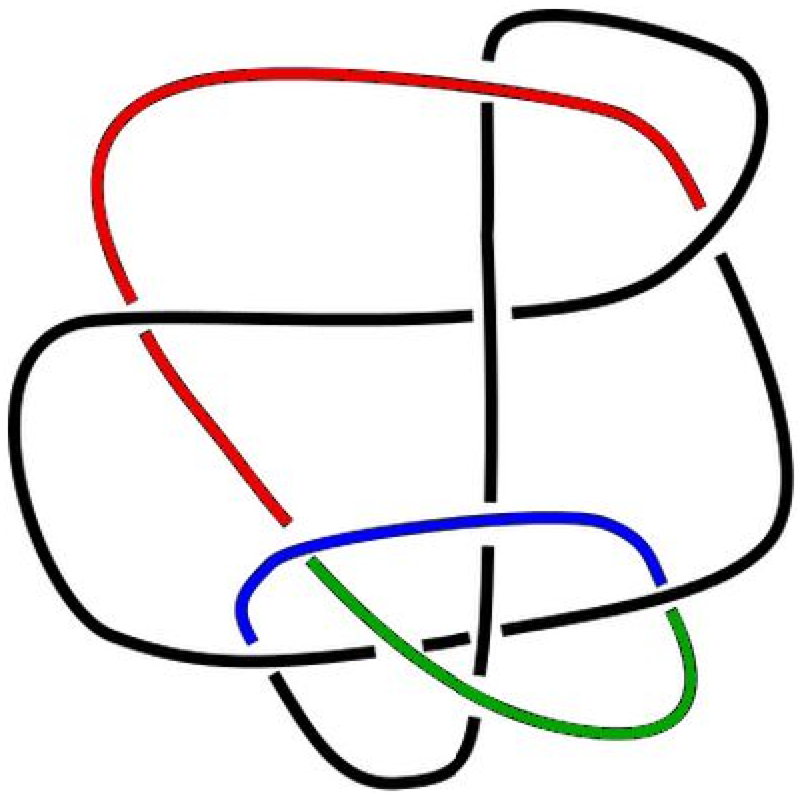
- at each crossing, the pattern is one of these:

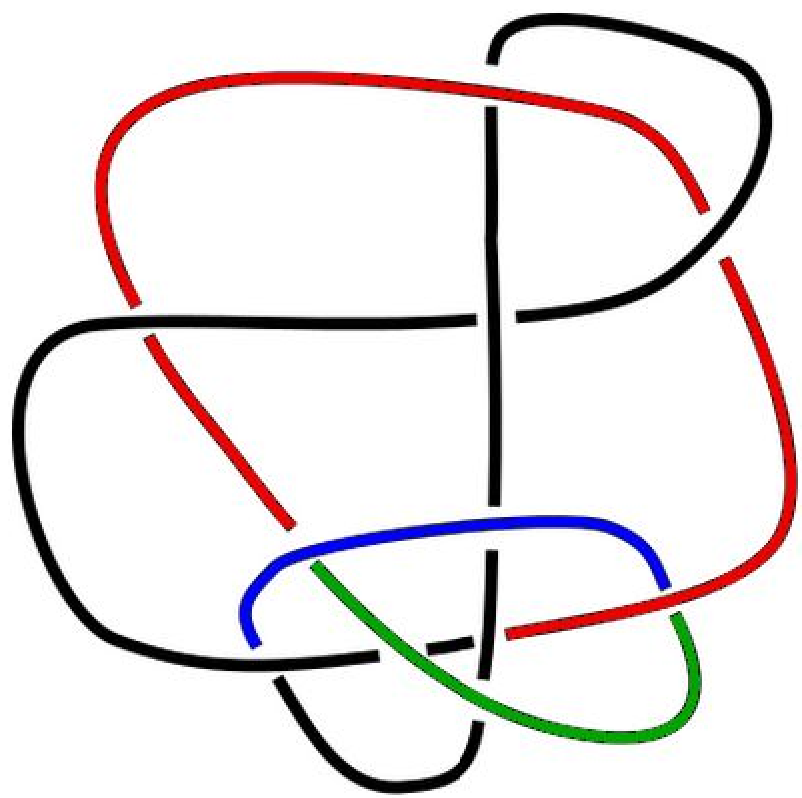


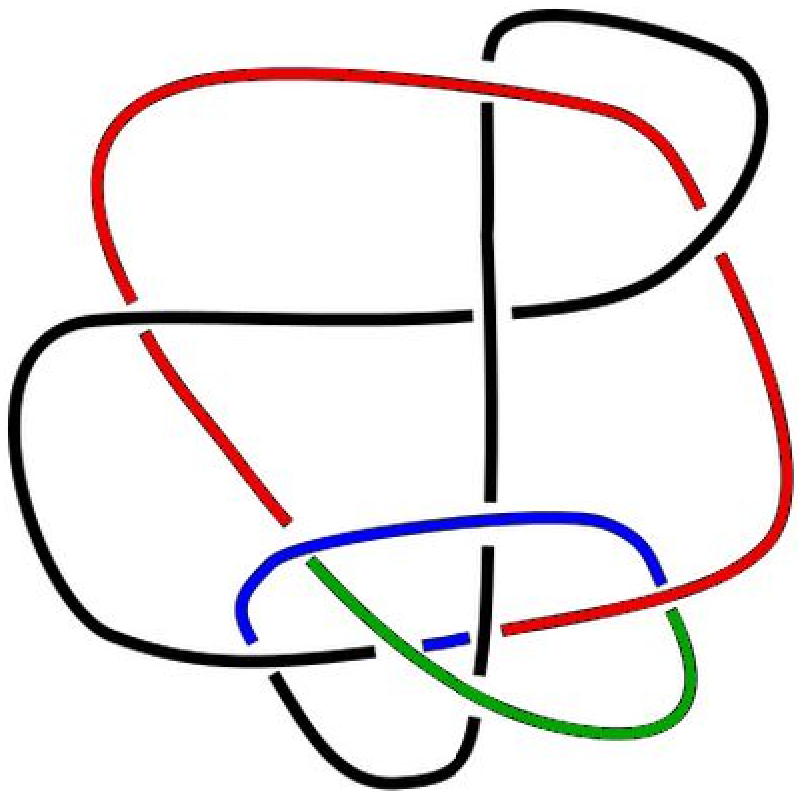


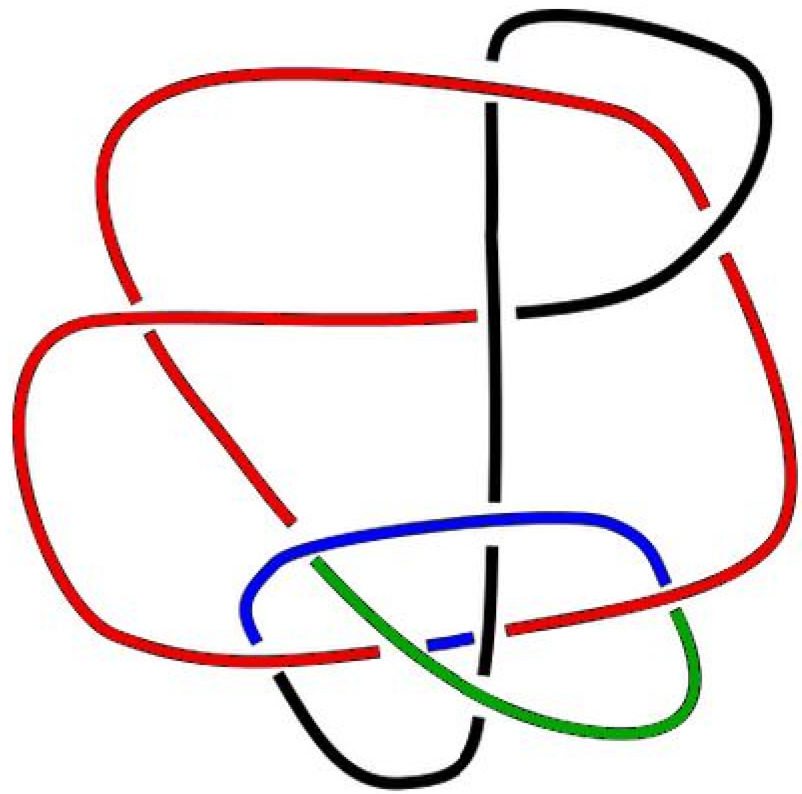


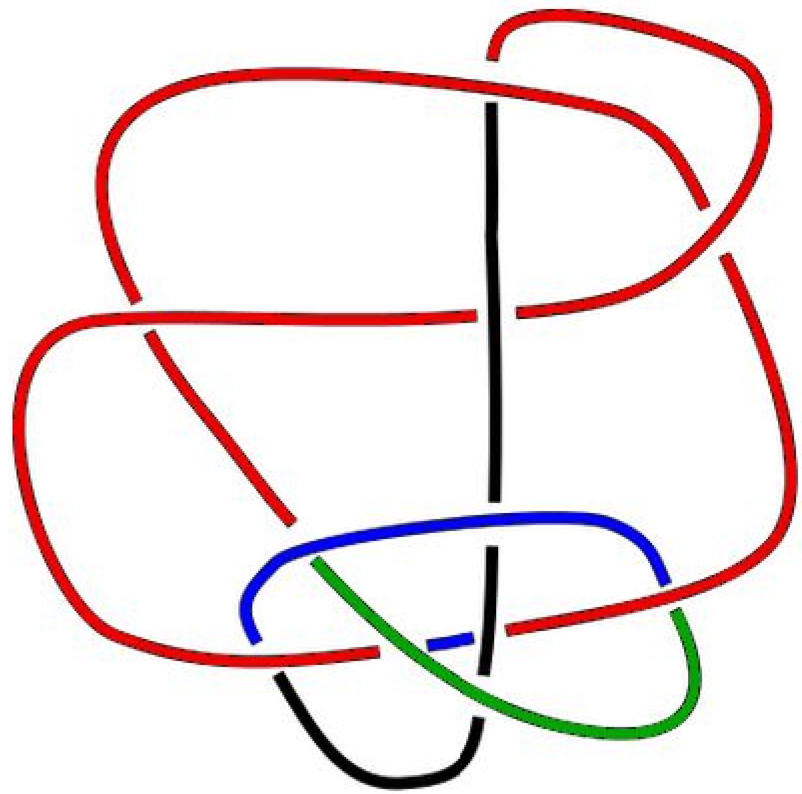


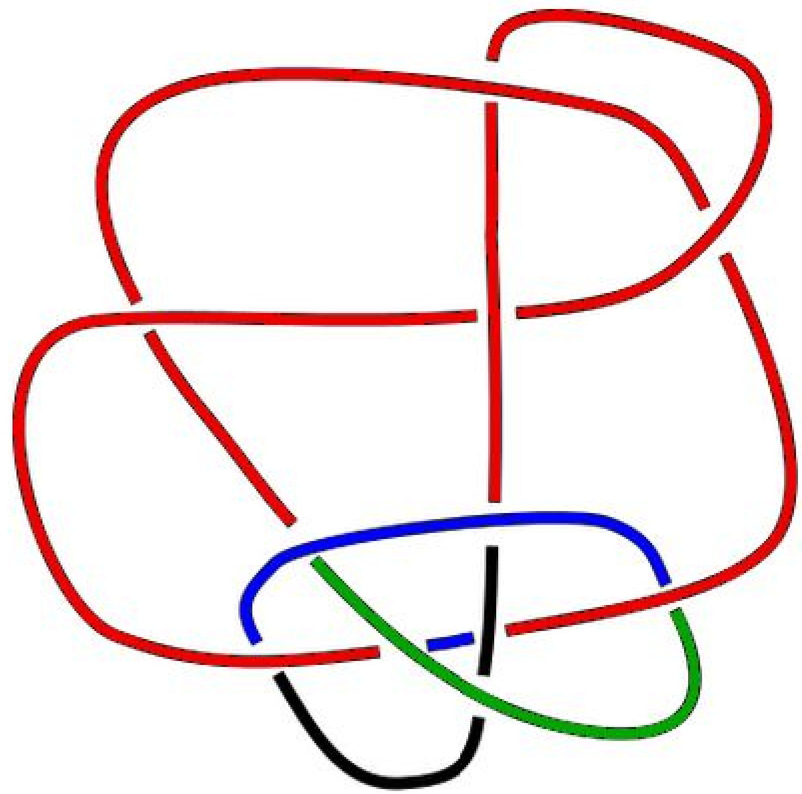


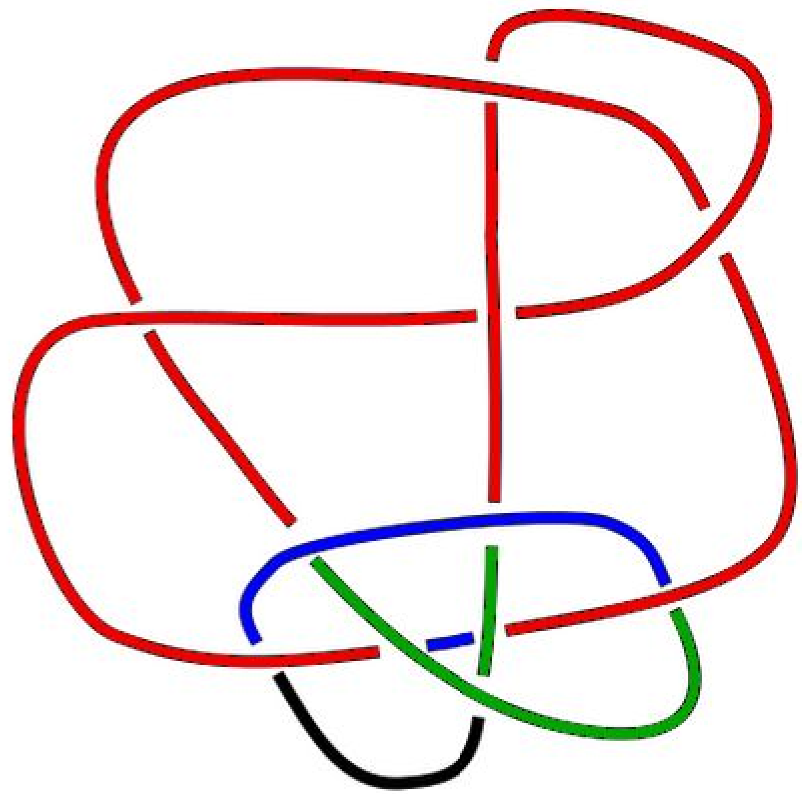


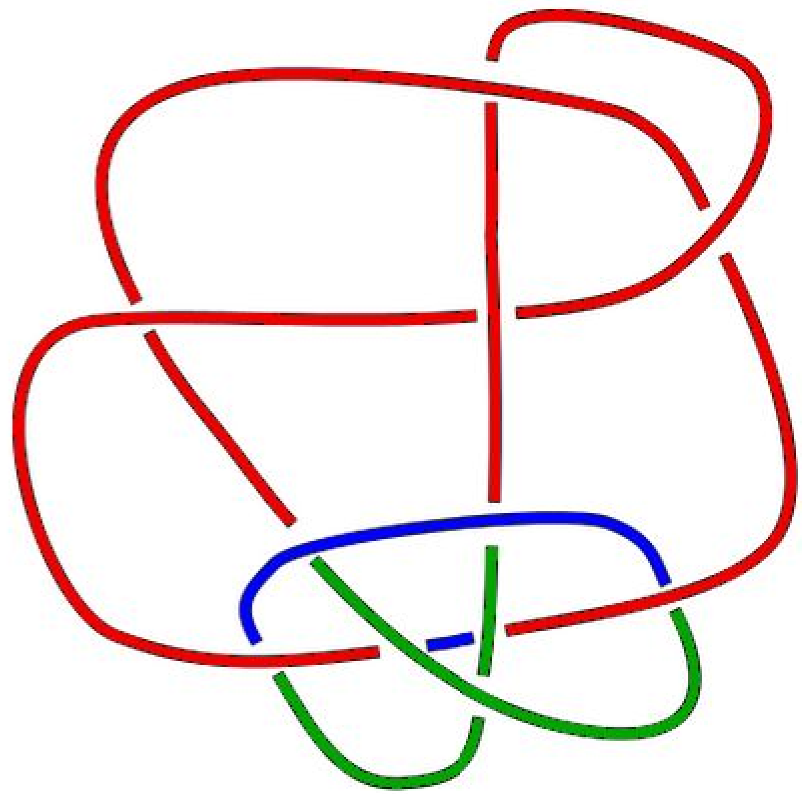




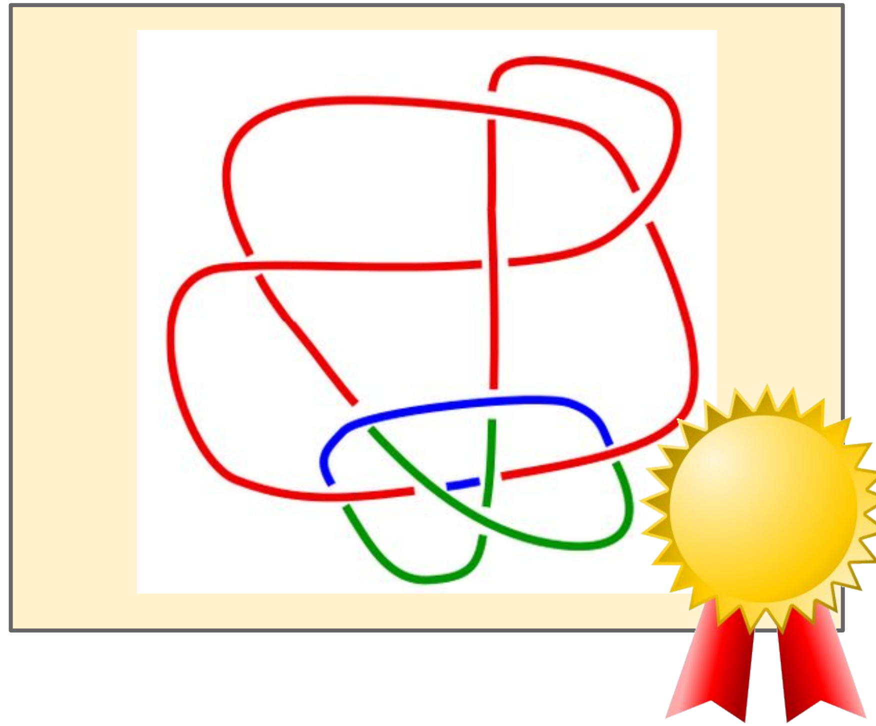


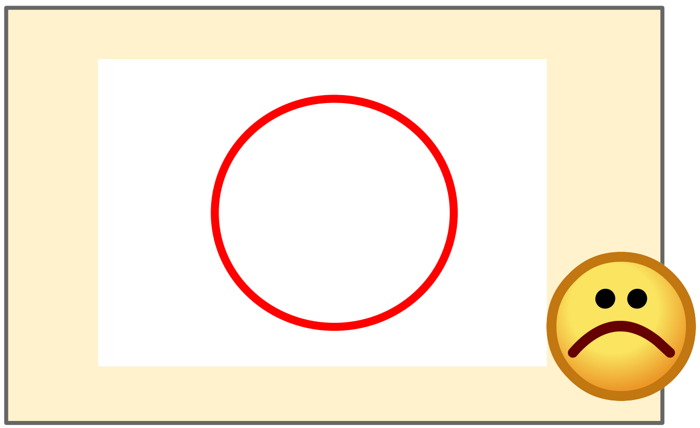


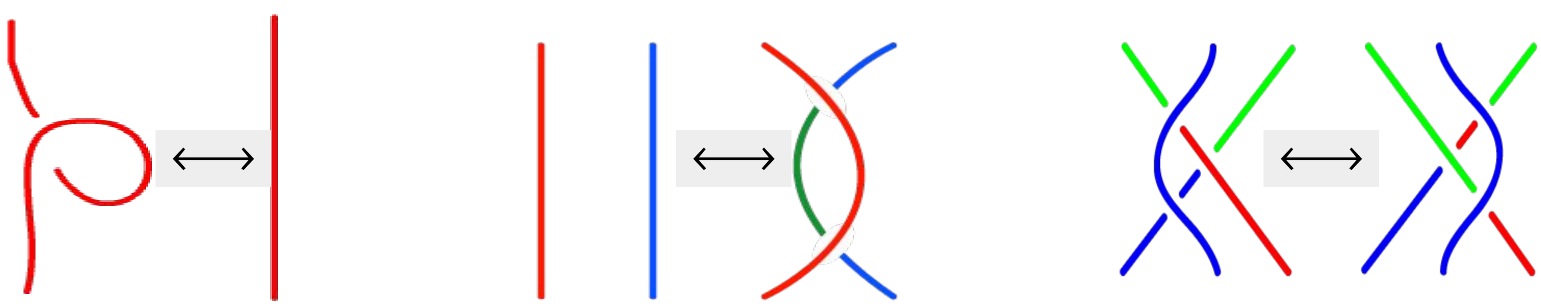




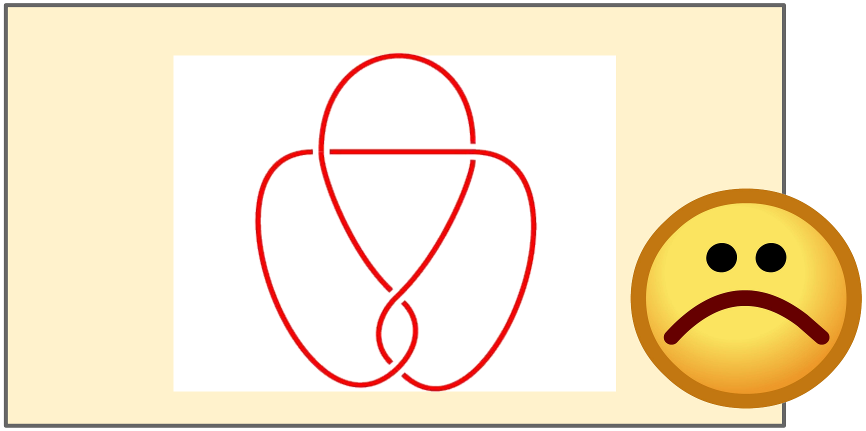
A tri-coloring is a "certificate of knottedness."

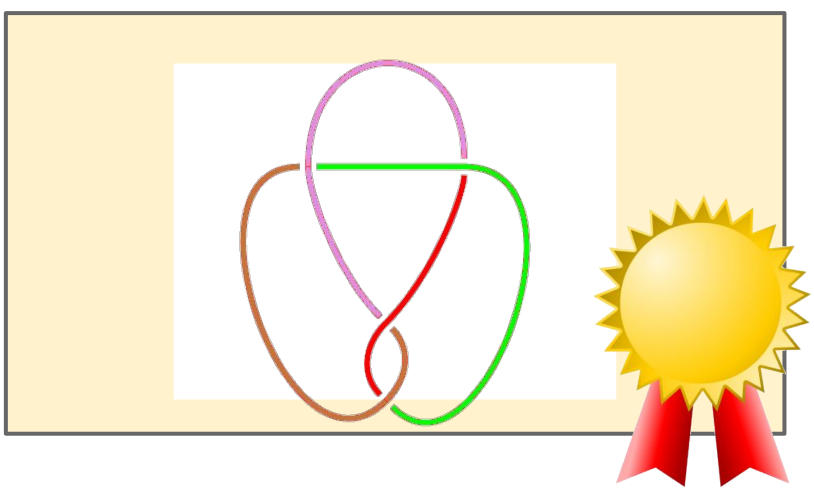






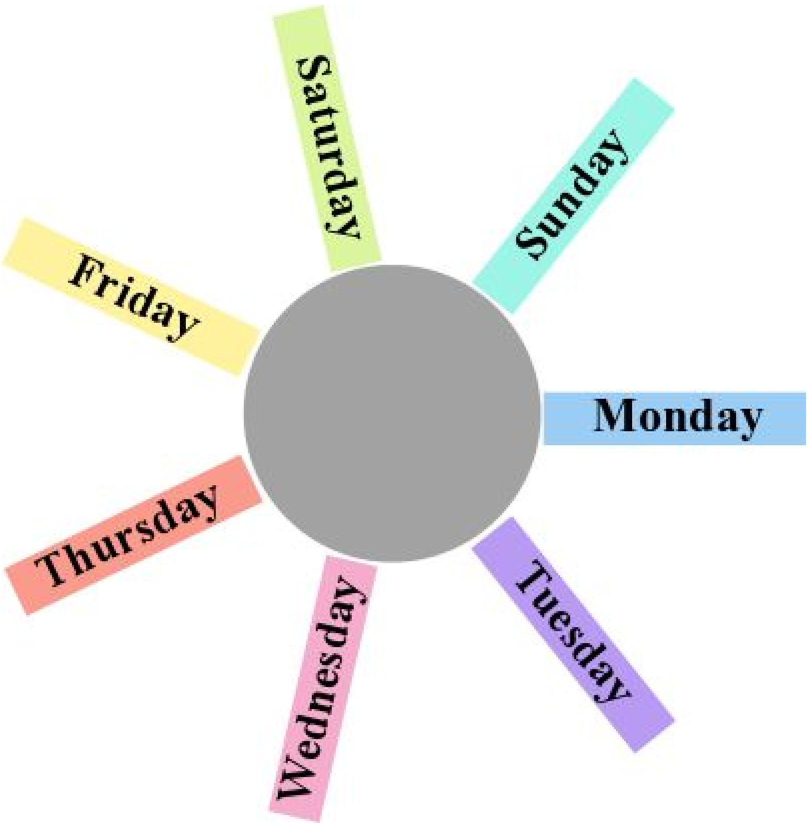
But some bona fide knots aren't tri-colorable

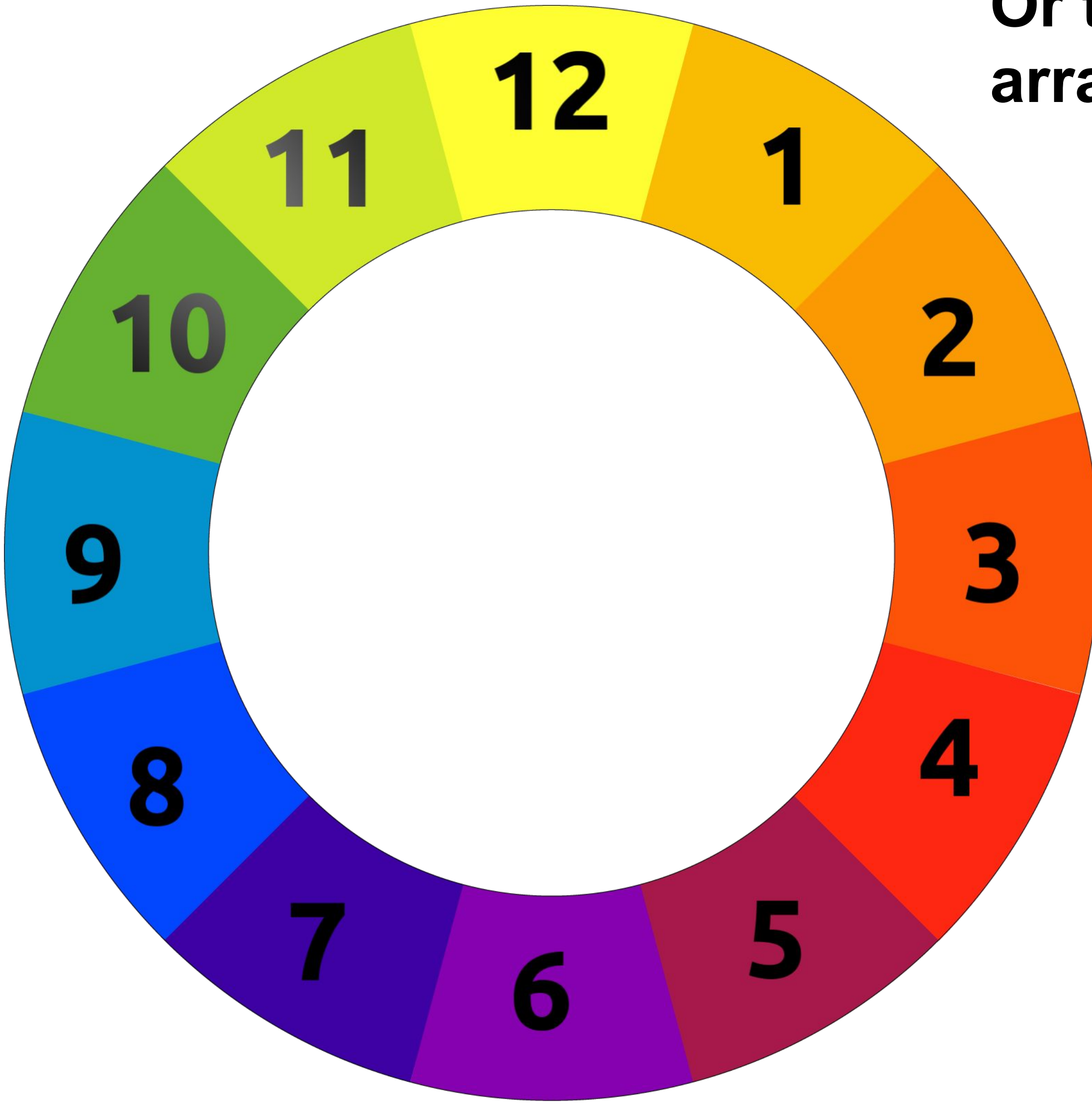




What about more colors?

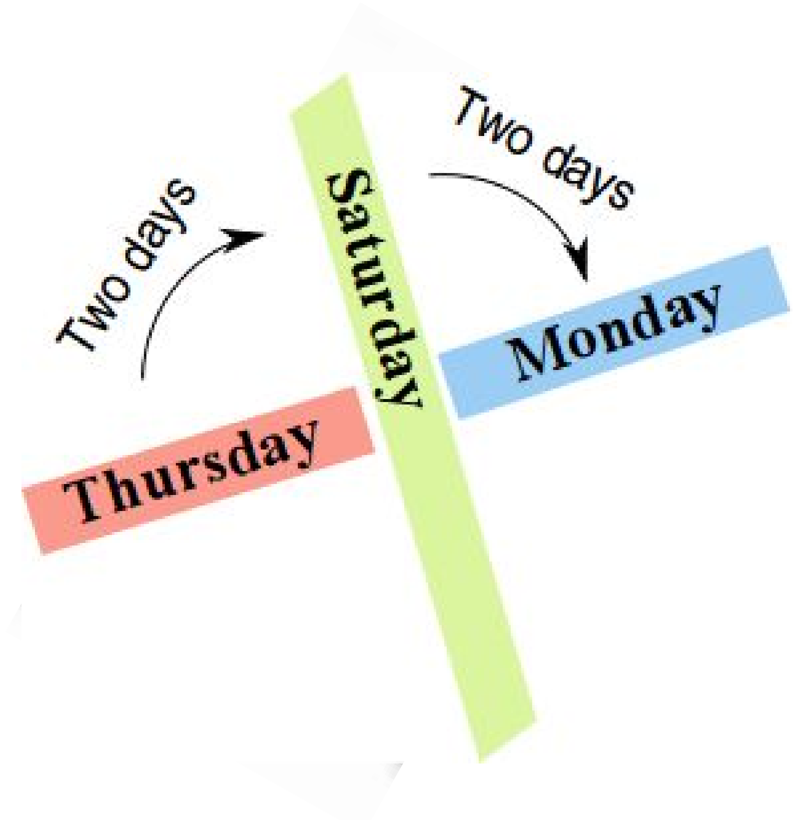
We can use seven colors, arranged in a circle ...

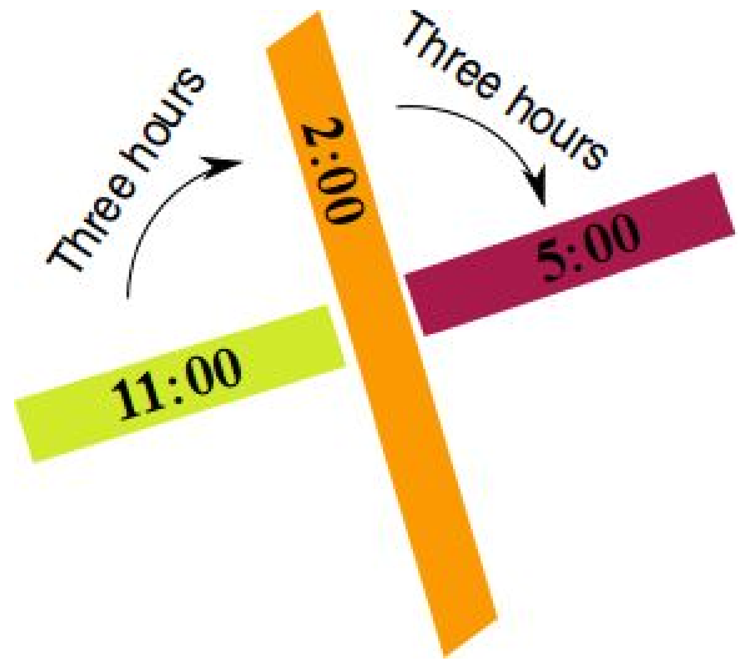


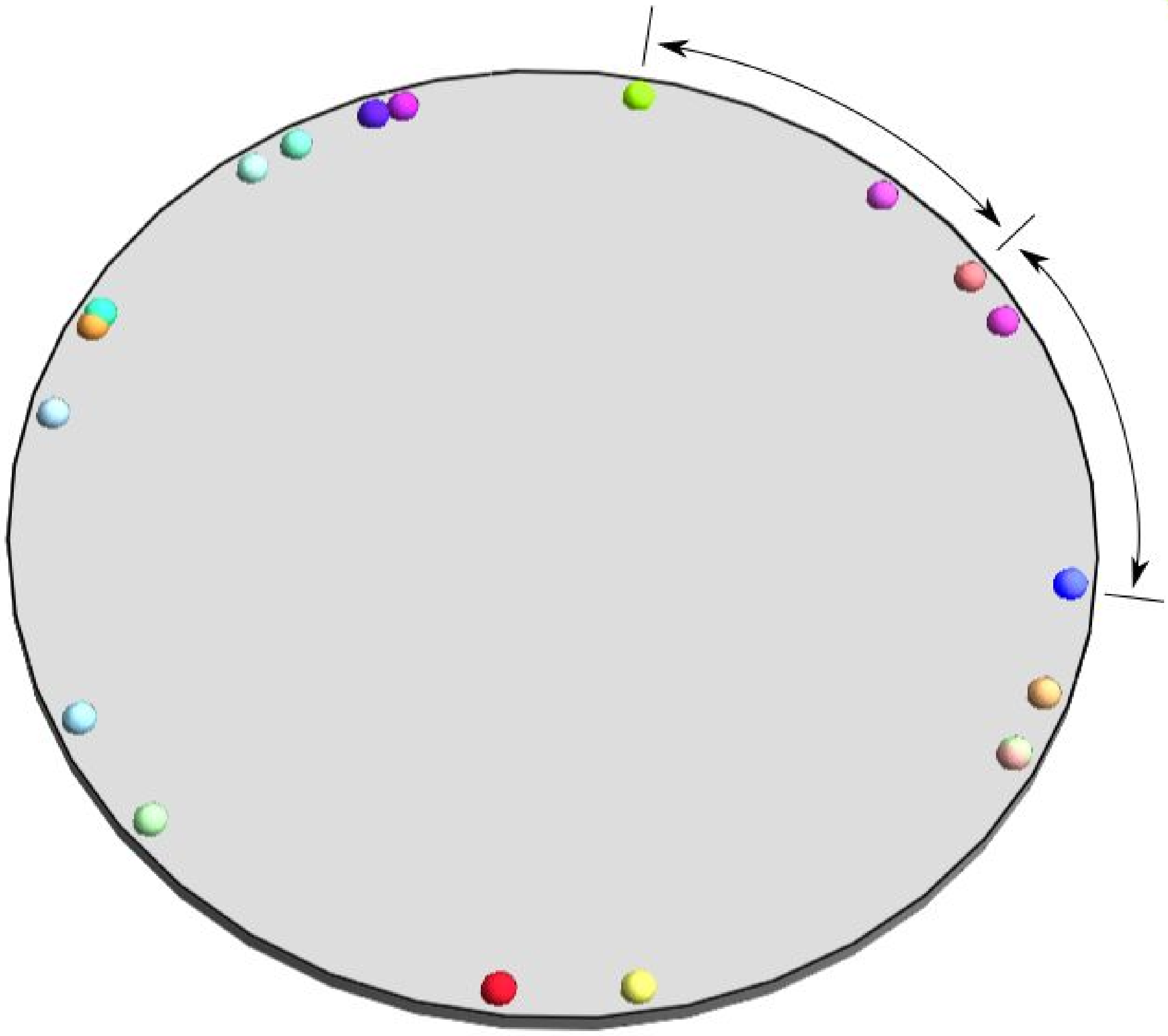


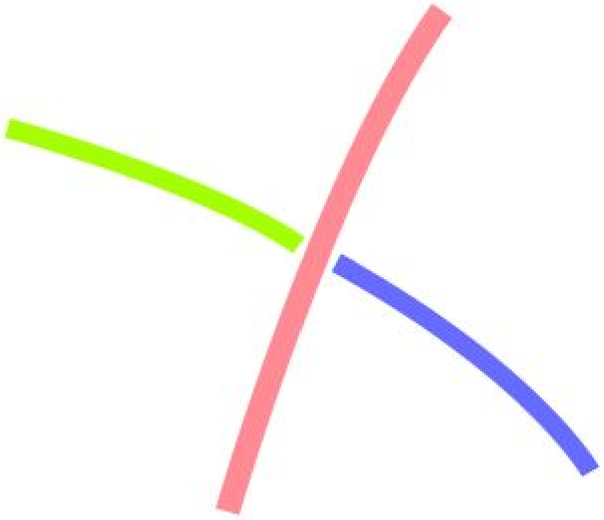
Or twelve colors, arranged in a circle …

Coloring rules





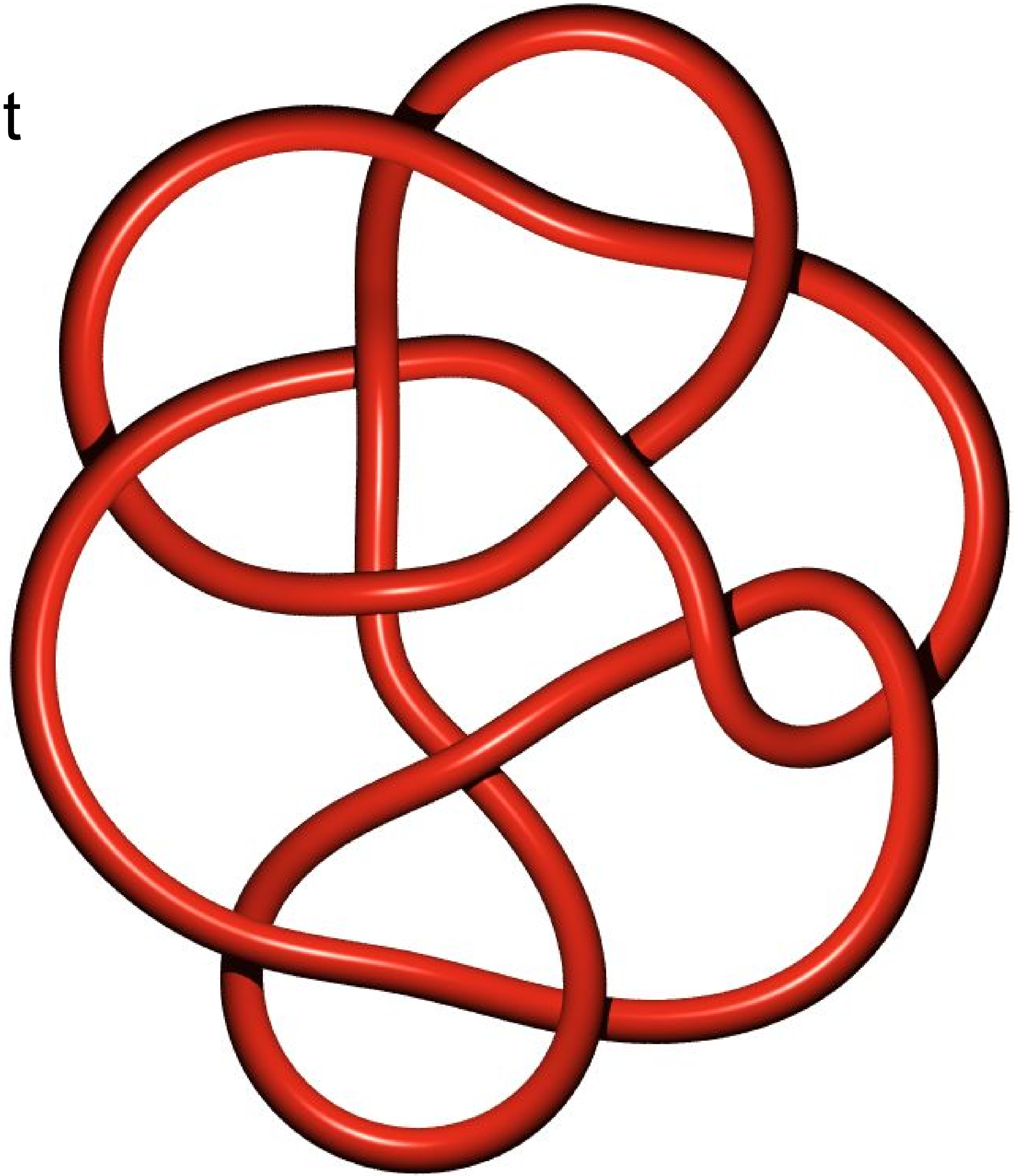




The Conway knot

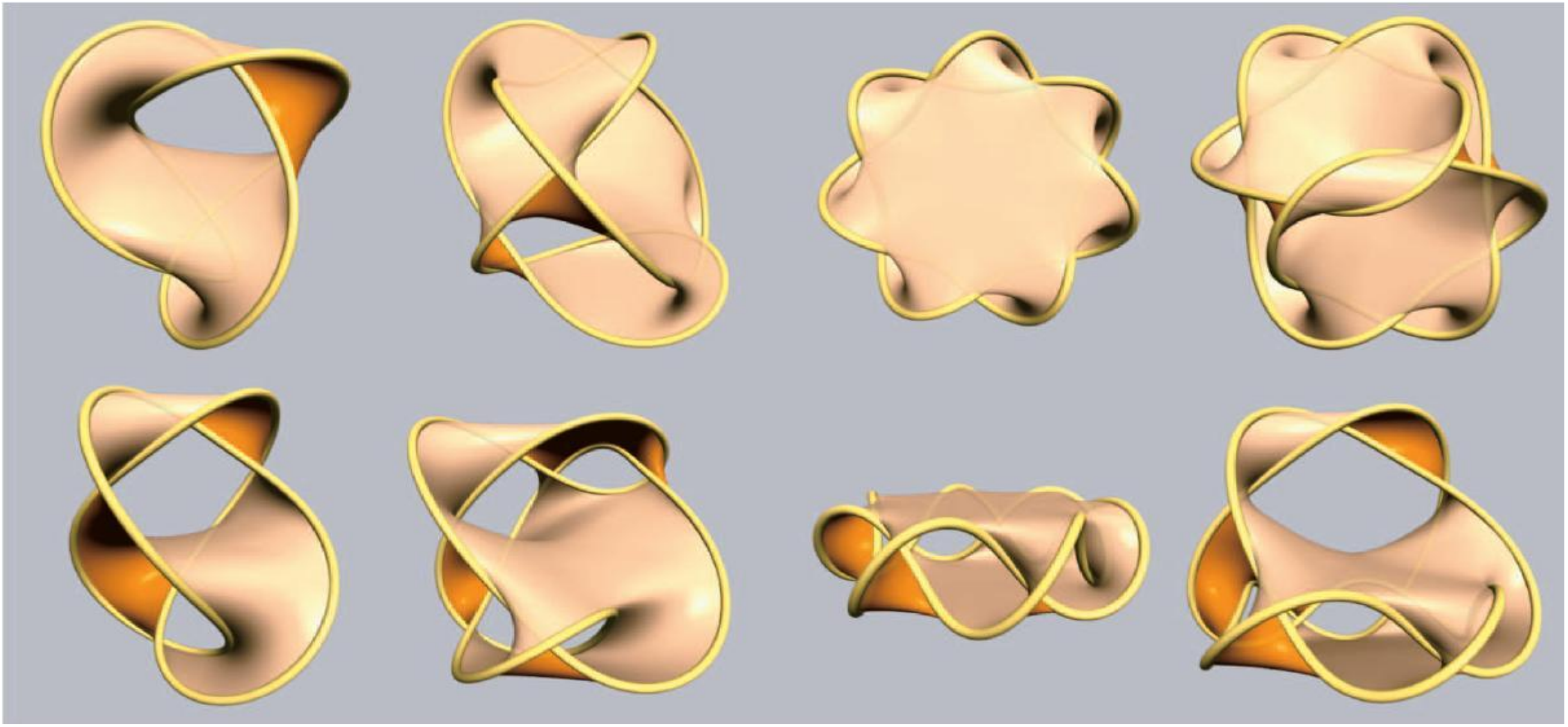
11n34

It only has the trivial coloring



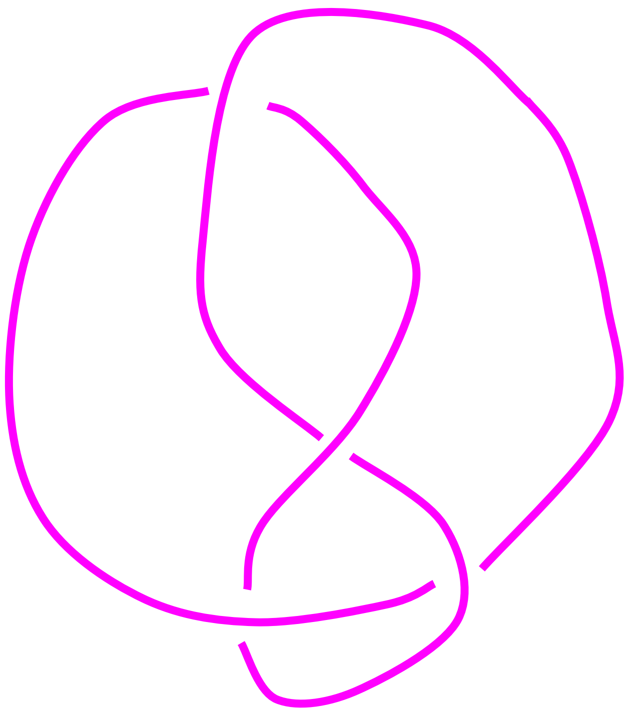
Other Invariants.

Every knots is the boundary of some oriented surface in 3-space.



Seifert's Algorithm.

1. Orient the knot.



Seifert's Algorithm.

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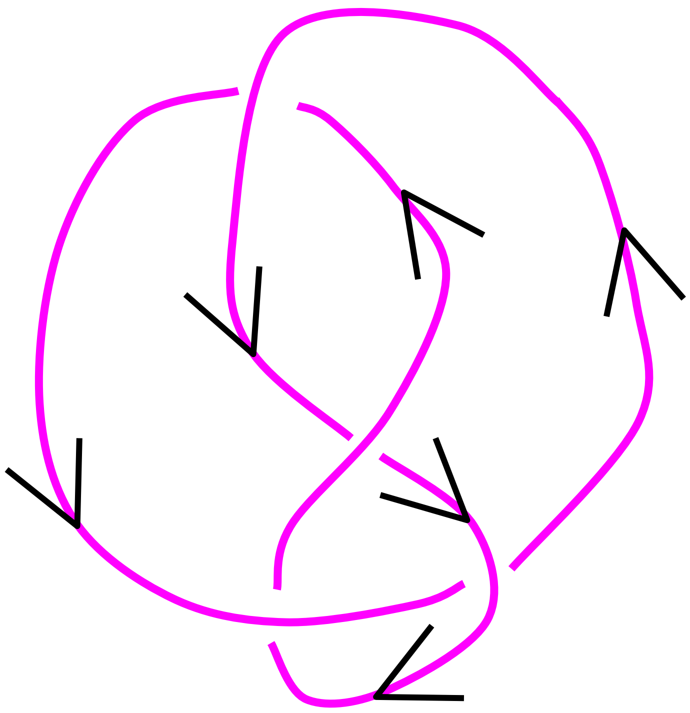
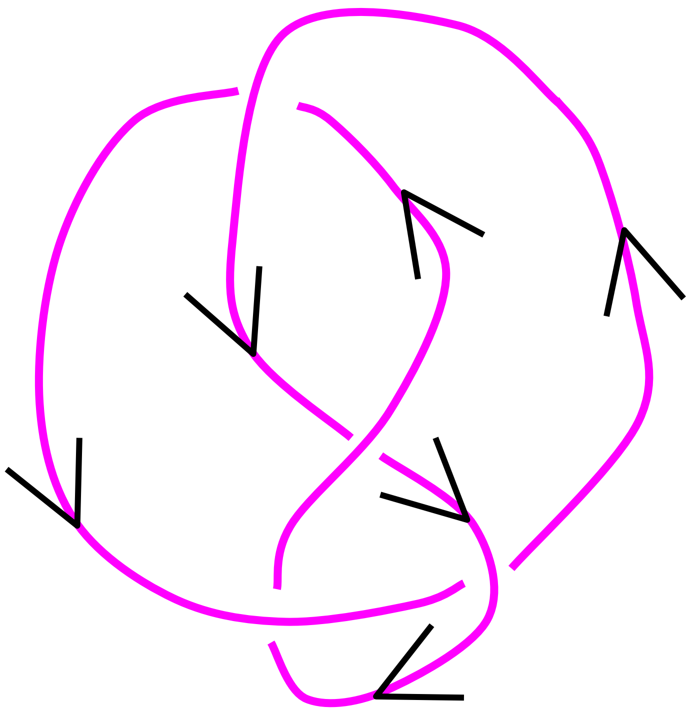


Seifert's Algorithm.

2. Resolve the crossing maintaining orientation.



Seifert's Algorithm.

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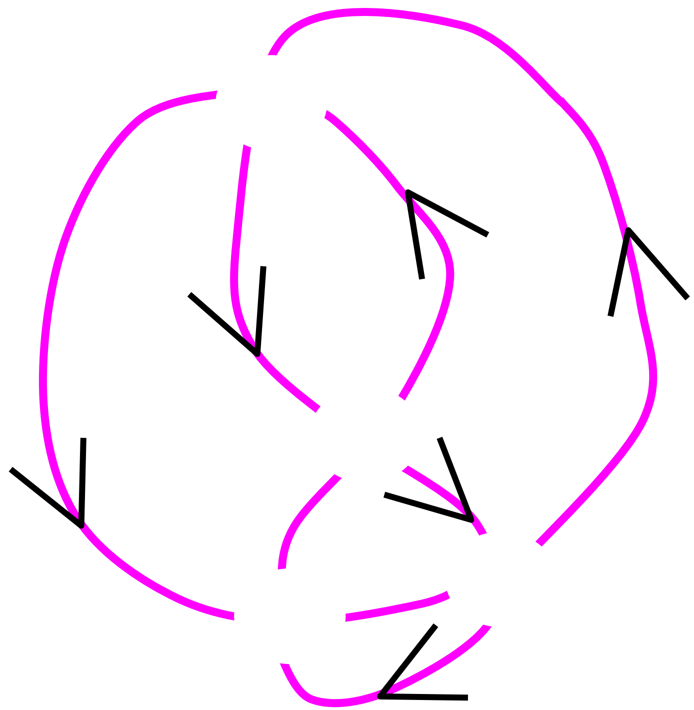

Seifert’s Algorithm.

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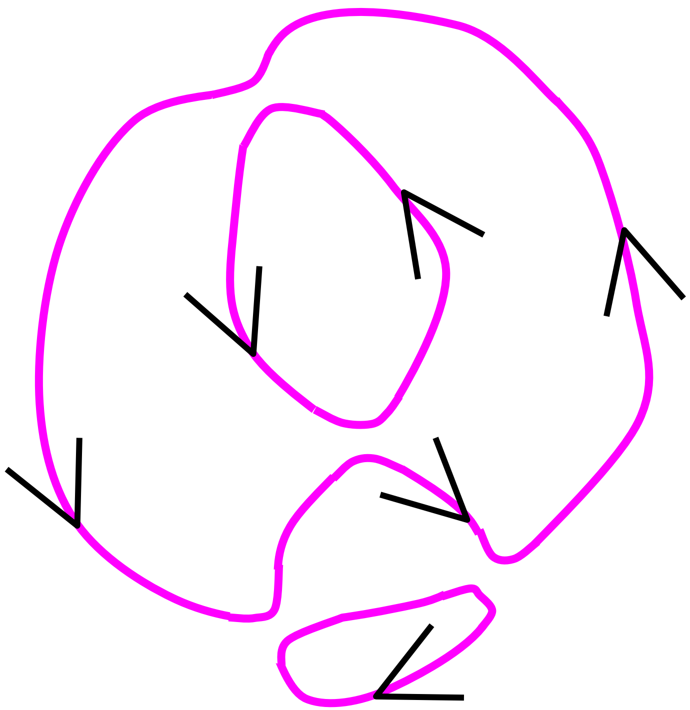
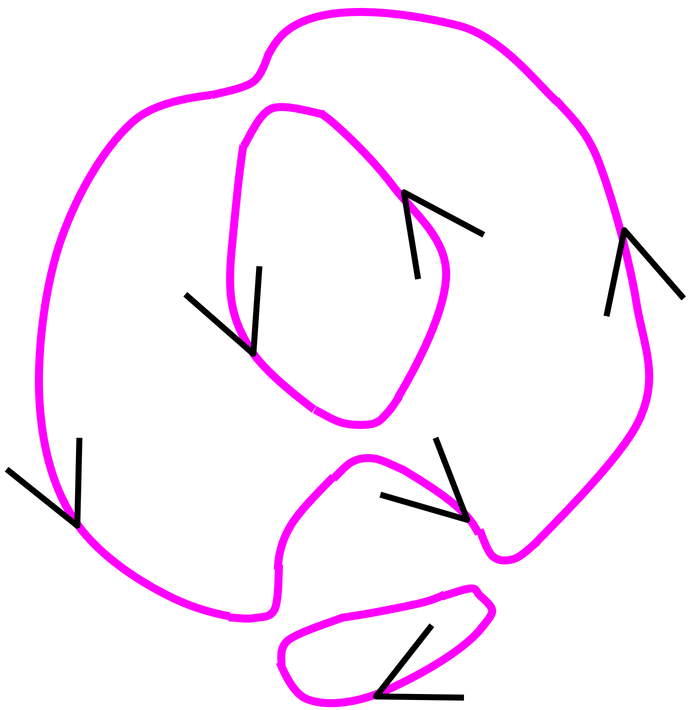


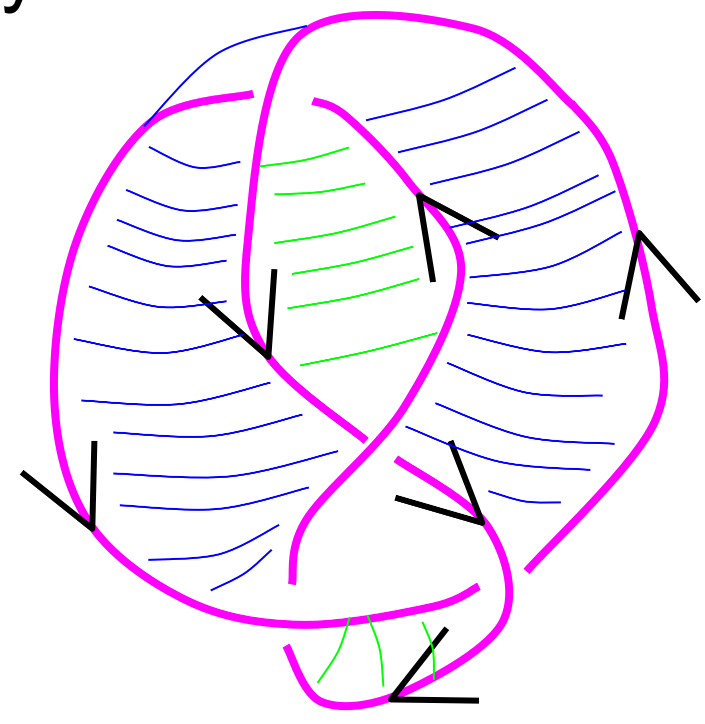
Seifert's Algorithm.

3. We now get a collection of nested oriented circles. Each bounds a disk in the plane.



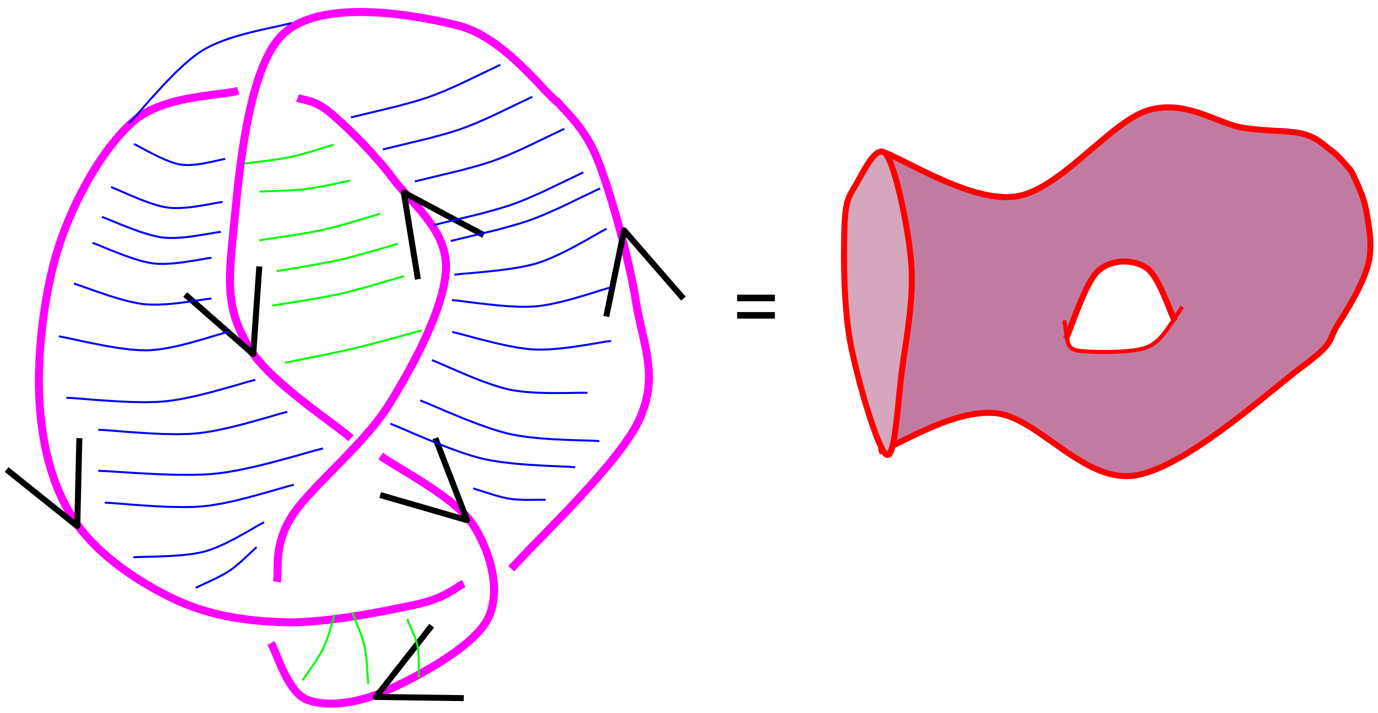
Seifert's Algorithm.

4. Add bands to these disks to match orientations and recover the knot as the boundary.



Seifert's Algorithm.

This surface is homeomorphic to a punctured torus.



Knot Genus

Recall that oriented surfaces are classified by the either Euler characteristic and the number of boundary components (assume the surface is connected). For closed surfaces this is an even integer written **2-2g** where **g** is the genus of the surface The Euler characteristic of Seifert surface of a knot is thus odd **1-2g.**

The genus of a knot is minimum g occurring for all surfaces (oriented embedded connected) bounding K.

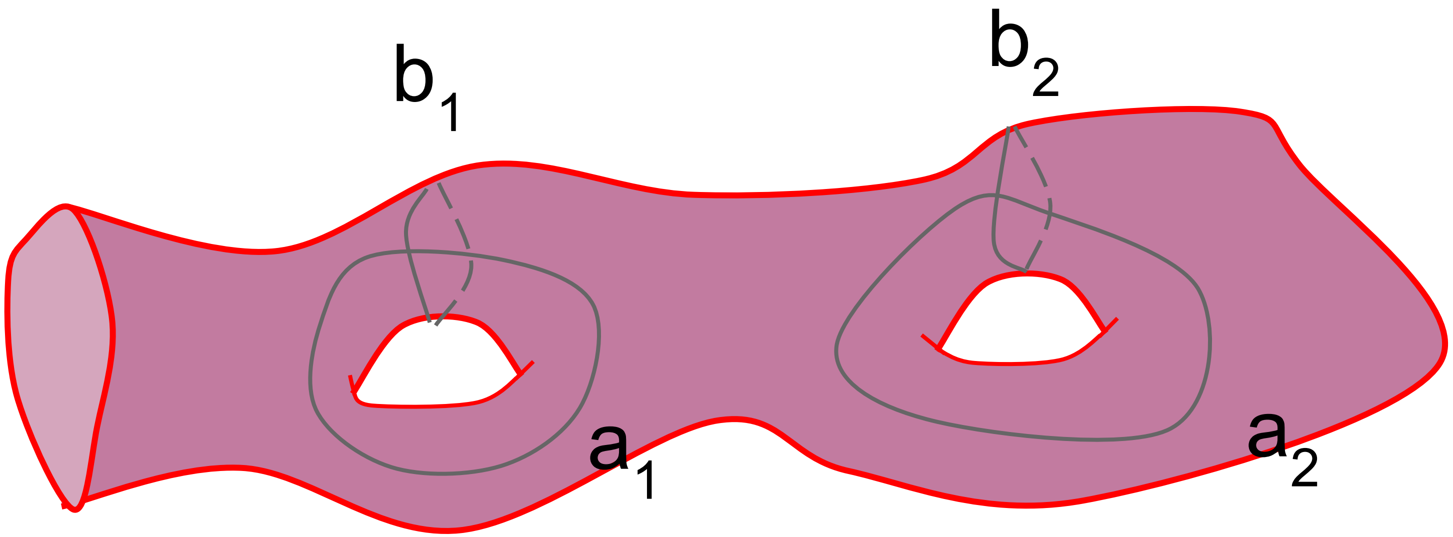
Certificate of Unknottedness

Note that K is an unknot if and only if K bounds a (smoothly) embedded disk i.e. has genus 0.

One direction is easy. The standard unknot bounds the standard disk. In the other direction if K bounds a disk. Shrink K along concentric circles in the disk until it is very small. Because the disk is smooth the new curve is basically a small ellipse.

The Seifert Matrix

Take the following collection of curves on a Seifert Surface.



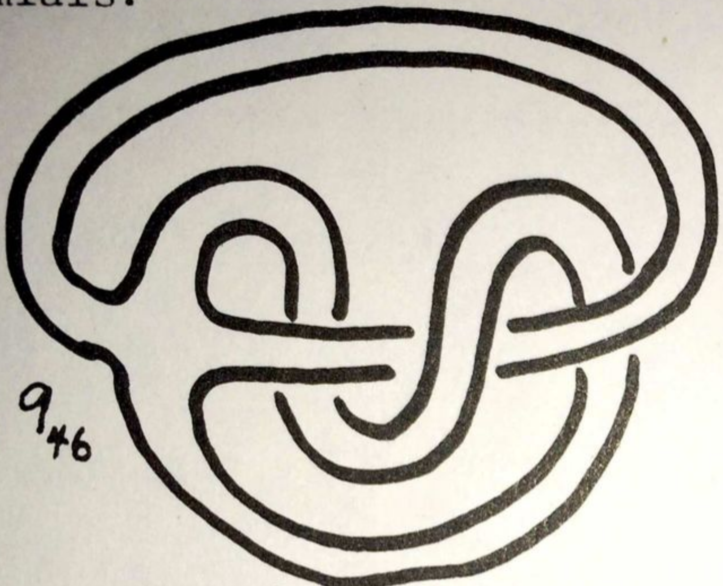
Push each of them off according to both the positive and negative normal to the surface.

We get a collection of curves in the complement of the Siefert surface $a^+_1, a^-_1, b^+_1, b^-_1$, etc. Form the matrix V whose entries are linking numbers of the minus curves with the plus curves.

Compute:

$$A_K(t)=\det(V-tV^T)$$

lthough the distinction is lost when we pass to Alexander ideals and olynomials.



9_{46}

$$V = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$V^T - tV = \begin{bmatrix} 0 & 1-2t \\ 2-t & 0 \end{bmatrix}$$

$$\Delta(t) = 2 - 5t + 2t^2$$

6_1

$$V = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

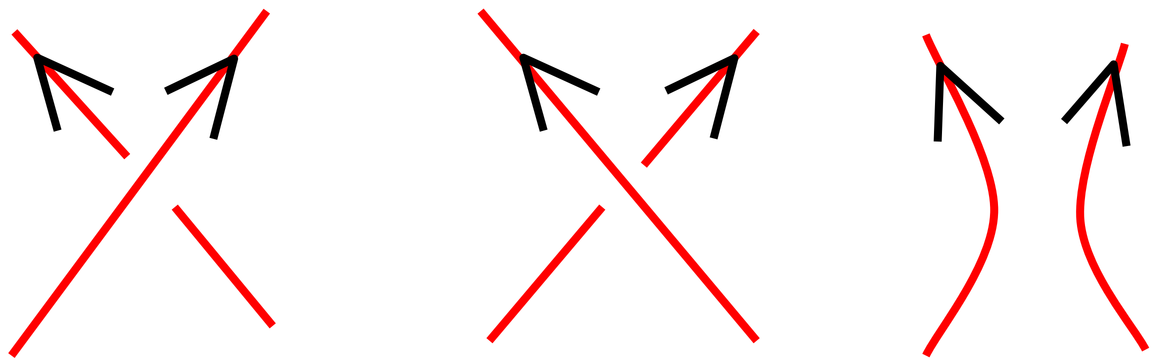
$$V^T - tV = \begin{bmatrix} 0 & 1-2t \\ 2-t & t-1 \end{bmatrix}$$

$$\Delta(t) = 2 - 5t + 2t^2$$

It turns out that this determinant, called the Alexander polynomial is a knot invariant up to multiplication by $+-t^n$. That is, it does not depend on the particular Siefert surface nor the way the curves used are drawn on that surface. Note that the spread in the degree of the Alexander polynomial gives a bound on the genus.

$$\tfrac{1}{2}\ \text{degree-spread}(\text{Alex}(K)) > \text{genus}(K)-1$$

Another computational tool for the Alexander polynomial. Skein relations.

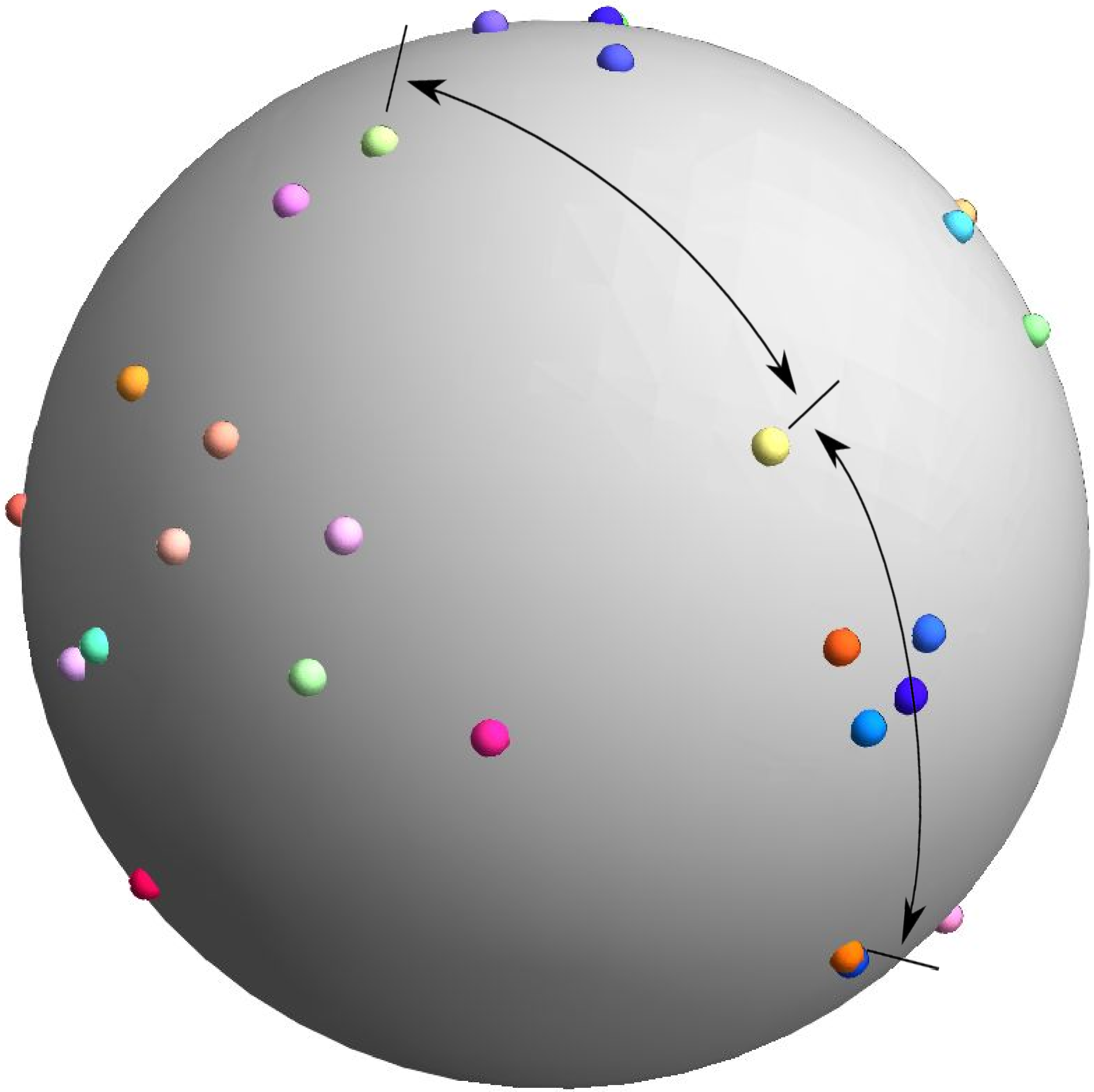


$t^{-1}A_{Kplus}(t)-tA_{Kminus}(t)=(t^{1/2}-t^{-½})A_{K0}(t)$

The Alexander Polynomial of a the Conway knots is 1. It can’t distinguish it from and unknots.

It turns out that a variant of coloring can detect non-trivial knotting reliably.

Two sphere colorings!

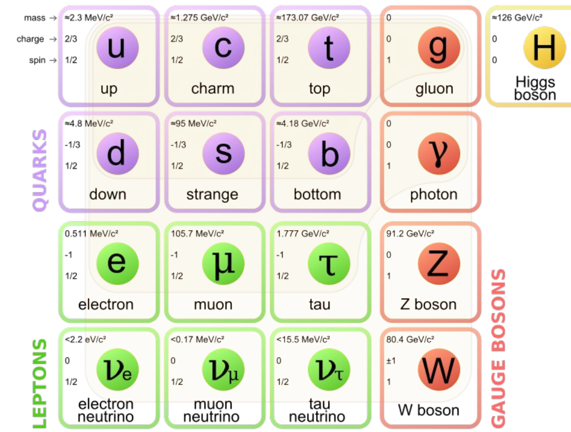


Sphere-colorings of knots

Theorem (Kronheimer-Mrowka, 2010):
Every knot except the unknot can be given a non-trivial sphere-coloring.

We can use colors arranged on the sphere to provide a certificate of knotedness for every bona-fide knot.

Techniques we use



- gauge theory
- Yang-Mills equations

Origins in theoretical physics, the description of fundamental particles (quarks, leptons) and their interactions

Karen Uhlenbeck

Cliff Taubes

Simon Donaldson