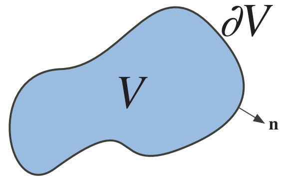

Figure 1: A volume V with a surface ∂V, and an outward unit normal vector $\mathbf{n}$ at each point on ∂V.

18.095 January 2016 exercises: Delta functions and distributions

1. In this part, you will consider the function $f(x) = \begin{cases} \ln|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$ and its (distributional) derivative, which is connected to something called the Cauchy Principal Value.

   (a) Show that $f(x)$ defines a regular distribution, by showing that $f(x)$ is locally integrable for all intervals $[a, b]$.

   (b) Consider the 18.01 derivative of $f(x)$, which gives $f'(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ \text{undefined} & x = 0 \end{cases}$. Suppose we just set "$f'(0) = 0$" at the origin to define $g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that this $g(x)$ is *not* locally integrable, and hence does not define a distribution.

   But the distributional derivative $f'\{\phi\}$ *must* exist, so this means that we have to do something different from the 18.01 derivative, and moreover $f'\{\phi\}$ is *not* a regular distribution. What is it?

   (c) Write $f\{\phi\} = \lim_{\epsilon \to 0^+} f_\epsilon\{\phi\}$ where $f_\epsilon\{\phi\} = \int_{-\infty}^{-\epsilon} \ln(-x)\phi(x)dx + \int_{\epsilon}^{\infty} \ln(x)\phi(x)dx$, since this limit exists and equals $f\{\phi\}$ for all ϕ from your proof in the previous part.[1] Compute the distributional derivative $f'\{\phi\} = \lim_{\epsilon \to 0^+} f'_\epsilon\{\phi\}$, and show that $f'\{\phi\}$ is precisely the *Cauchy Principal Value* (google the definition, e.g. on Wikipedia) of the integral of $g(x)\phi(x)$.

2. In class, we only looked explicitly at 1d distributions, but a distribution in d dimensions $\mathbb{R}^d$ can obviously be defined similarly, as maps $f\{\phi\}$ from smooth localized functions $\phi(\mathbf{x})$ to numbers. Analogous to class, define the distributional gradient ∇f by $\nabla f\{\phi\} = f\{-\nabla\phi\}$.

   Consider some finite volume V with a surface ∂V, and assume ∂V is differentiable so that at each point it has an outward-pointing unit normal vector $\mathbf{n}$, as shown in figure 1. Define a "surface delta function" $\delta(\partial V)\{\phi\} = \oint_{\partial V} \phi(\mathbf{x}) d^{d-1}\mathbf{x}$ to give the surface integral $\oint_{\partial V}$ of the test function.

   Suppose we have a regular distribution $f\{\phi\}$ defined by the function $f(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) & \mathbf{x} \in V \\ f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases}$, where we may have a discontinuity $f_2 - f_1 \neq 0$ at ∂V.

   (a) Show that the distributional gradient of f is

   $$\nabla f = \delta(\partial V)\,[f_1(\mathbf{x}) - f_2(\mathbf{x})]\,\mathbf{n}(\mathbf{x}) + \begin{cases} \nabla f_1(\mathbf{x}) & \mathbf{x} \in V \\ \nabla f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases},$$

   where the second term is a regular distribution given by the ordinary 18.02 gradient of f_1 and f_2 (assumed to be differentiable), while the first term is the singular distribution

   $$\delta(\partial V)\,[f_1(\mathbf{x}) - f_2(\mathbf{x})]\,\mathbf{n}(\mathbf{x})\{\phi\} = \oint_{\partial V} [f_1(\mathbf{x}) - f_2(\mathbf{x})]\,\mathbf{n}(\mathbf{x})\phi(\mathbf{x}) d^{d-1}\mathbf{x}.$$

   You can use the integral identity that $\int_V \nabla\psi d^d\mathbf{x} = \oint_{\partial V} \psi \mathbf{n} d^{d-1}\mathbf{x}$ to help you integrate by parts.

[1] More explicitly, note that $f\{\phi\} - f_\epsilon\{\phi\} = \int_{-\epsilon}^{\epsilon} \ln|x|\phi(x)dx \leq (\max|\phi|) \int_{-\epsilon}^{\epsilon} \ln|x|dx \to 0$; you should have done the something like the last integral explicitly in the previous part.