Steady states of coupled dynamical systems
(IAP lecture series)

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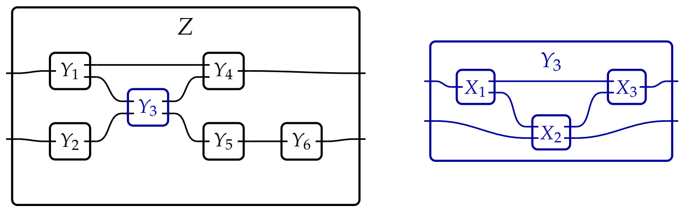
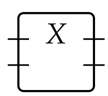
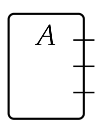
1. Draw the diagram that results from substituting the right into the left.



2.a. Consider the box below. What are X^{in} and X^{out} as sets?

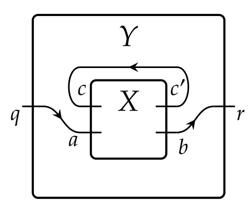


2b. Consider the box below. What are A^{in} and A^{out} as sets?



2c. Write down the set-theoretic formula for the following wiring diagram, i.e., a pair of functions

$$X^{\text{in}} \to X^{\text{out}} + Y^{\text{in}} \qquad \text{and} \qquad Y^{\text{out}} \to X^{\text{out}}.$$



3a. Draw the box X_1 with $X_1^{\text{in}} = \{a, b\}$ and $X_1^{\text{out}} = \{c\}$.

3b. Draw the box X_2 with $X_2^{\text{in}} = \{d, e\}$ and $X_2^{\text{out}} = \{f, g\}$.

3c. Let Y be the box $Y^{\text{in}} = \{h, i\}$ and $Y^{\text{out}} = \{j, k\}$. Draw the wiring diagram that corresponds to the following pair of functions

$X_1^{\text{in}} + X_2^{\text{in}}$	$\longrightarrow$	$X_1^{\text{out}} + X_2^{\text{out}} + Y^{\text{in}}$	Y^{out}	$\longrightarrow$	$X_1^{\text{out}} + X_2^{\text{out}}$
a		h	j		c
b		g	k		f
d		c			
e		i			

Hint: it has two inner boxes, X_1 and X_2, and outer box Y. You will find that at least one wire splits.

4. Let $A = \{\mathrm{T}, \mathrm{F}\}$ and $B = \{\mathrm{Red}, \mathrm{Green}, \mathrm{Blue}\}$. Below is a small example of an (A, B)-discrete system (i.e., a possible inhabitant of the box $A \to \square \to B$), shown both in tabular form and as a transition diagram.

Input	State	Readout	Next state
T	1	Blue	2
F	1	Blue	1
T	2	Red	2
F	2	Red	3
T	3	Green	4
F	3	Green	4
T	4	Blue	1
F	4	Blue	4

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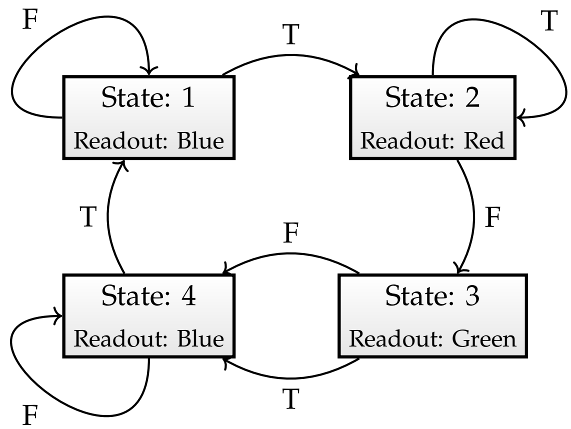


(1)

a. Write down the $A \times B$ steady state matrix for this system.

b. Let $C = \{\mathrm{Right}, \mathrm{Left}, \mathrm{Up}, \mathrm{Down}\}$. Make up your own (B, C)-dynamical system, such that it has two states. Show it in both table form (as to the left above) and transition-diagram form (as to the right above). Then write down its steady-state matrix.

c. Take the dynamical systems from parts a and b, and compose them in series. Write down your result as a transition diagram.

d. Multiply the steady-state matrices and see if the result agrees with the steady state matrix of the composed system.

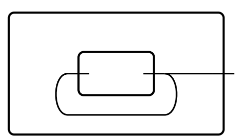
e. Write down the steady state matrix that would arise if the dynamical systems from parts a and b were composed in parallel.

5. Consider the following open continuous dynamical system (inhabiting the box $A \,\boxed{\ }\, B$). Here $a \in A$ is an input parameter and $b \in B$ is the readout.

$$
\begin{aligned}
\dot{x}_1 &= x_1^2 - a \\
\dot{x}_2 &= x_1 - x_2 \\
b &= x_2
\end{aligned}
$$

a. Draw the bifurcation diagram for this system in the plane $\mathbb{R}^2$, with A as the horizontal axis and B as the vertical axis. That is, draw a point at (a, b) if and only if there exists a steady state (x_1, x_2) when the parameter is a, such that the output is b.

b. Now suppose that the output B is fed back into the input A, as in the following wiring diagram:



The bifurcation diagram is now just some points on the b axis, because there are no input parameters to the system. Draw the bifurcation diagram, being sure to indicate the values of these points.