18.095–IAP Mathematics Lecture Series: Overdamped dynamics of small objects in fluids

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Each subproblem counts 20%.

Problem 1: Sedimentation

Assume a spherical particle of mass (radius a, mass density ρ) moves vertically under the influence of gravitational acceleration $\boldsymbol{g} = -g\boldsymbol{e}_z$ in a fluid (mass density ρ_f, dynamic viscosity μ). The initial position of the particle at time $t = 0$ is $\boldsymbol{r}(0) = \boldsymbol{0}$ and its velocity $\boldsymbol{v}(0) = u_0\boldsymbol{e}_z$. Neglecting thermal fluctuations:

(i) Calculate the settling speed and estimate the associated Reynolds number, defined as the ratio of inertial and viscous forces

(ii) Find the particle's position $\boldsymbol{r}(t)$ and velocity $\boldsymbol{v}(t)$ at time $t > 0$.

(iii) Compare the stationary speed for spheres of radius $a = 1$mm and $a = 1\mu$m consisting of gold, plastic and air, respectively, in water. How long does it typically take for those particles to lose memory of their initial velocity? What can you say about the validity of these results?

Problem 2: Over-damped Brownian motion

(i) Show that the formal solution of the inhomogeneous ODE

$$m\ddot{\boldsymbol{x}} = -\gamma\dot{\boldsymbol{x}} + \boldsymbol{f}(t) \tag{1}$$

with initial conditions $\boldsymbol{x}(0) = 0$ and $\dot{\boldsymbol{x}}(0) = \boldsymbol{v}_0$ is given by

$$\boldsymbol{x}(t) = \boldsymbol{v}_0\tau\left(1 - e^{-t/\tau}\right) + \frac{1}{m}e^{-t/\tau}\int_0^t du\, e^{u/\tau}\int_0^u ds\, \boldsymbol{f}(s) \tag{2}$$

(ii) Now consider the over-damped zero-Reynolds number limit, where inertia can be neglected, $m\ddot{\boldsymbol{x}} = 0$, and assume that $\boldsymbol{f}$ depends on some hidden parameter λ, so that

$$\gamma\dot{\boldsymbol{x}} = \boldsymbol{f}(t;\lambda) \tag{3}$$

Show that in this limit case

$$\boldsymbol{x}(t;\lambda) = \boldsymbol{x}(0) + \frac{1}{\gamma}\int_0^t ds\, \boldsymbol{f}(s;\lambda) \tag{4}$$

Consider an abstract averaging procedure $\langle\cdot\rangle$ over the hidden parameter λ that is linear[1], commutes with time-integration and satisfies

$$\langle 1\rangle = 1\,, \qquad \langle \boldsymbol{f}(s;\lambda)\rangle = 0\,, \qquad \langle \boldsymbol{f}(s;\lambda)\cdot\boldsymbol{f}(t;\lambda)\rangle = 2D\,\delta(t-s) \tag{5}$$

where D is a constant. Compute the mean position $\langle \boldsymbol{x}(t;\lambda)\rangle$ and the mean squared displacement $\langle[\boldsymbol{x}(t;\lambda)]^2\rangle$.

*If you have questions, please email me (`dunkel@math.mit.edu`).

[1] This means that $\langle a\boldsymbol{f}(\lambda) + b\boldsymbol{g}(\lambda)\rangle = a\langle\boldsymbol{f}(\lambda)\rangle + b\langle\boldsymbol{g}(\lambda)\rangle$ for all real numbers a, b