ON THE CONVERGENCE OF SERIES

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(1) Show that the ratio test is inconclusive when applied to $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (in fact it is inconclusive when applied to any series of the form $\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$, where P and Q are polynomials in n.)

(2) Use the ratio test to show that the series $\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for every real x. This is the exponential function on the real line. Given any $d > 0$, show that
$$\frac{\exp(x)}{x^d} \to \infty$$
as $x \to \infty$ (this shows the general fact that "exponentials beat polynomials").

(3) Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n \log n}$ diverges but that $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^{1+\epsilon}}$ converges for any $\epsilon > 0$.

(4) Let A denote the set of all natural numbers that do not contain a '42' in their decimal expansion. For example $19,756,583,920$ is in A but $92,835,934,274,850$ is not in A. Does $\sum_{n \in A} \frac{1}{n}$ converge or diverge?

(5) Define
$$e := \exp(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}.$$
We'll show that e is irrational. Assume, for the sake of contradiction that e can be expressed as a fraction a/b, *i.e.*
$$\frac{a}{b} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots.$$
Multiplying both sides by $b!$ we have that
$$b!\frac{a}{b} = \sum_{n=1}^{b} \frac{b!}{n!} + \sum_{n=b+1}^{\infty} \frac{b!}{n!}.$$
The left-hand side is clearly an integer (it is equal to $(b-1)!a$) and the first term on the right-hand side is also an integer. This means that the second term on the right-hand side,
$$x = \sum_{n=b+1}^{\infty} \frac{b!}{n!},$$
is a positive integer. Bound x above by a convergent geometric series to show that $x < 1$ and hence derive a contradiction that proves the irrationality of e.

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