On The Convergence of Series
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Contents

1 Geometric Series

In mathematics an infinite series is a sum of infinitely many numbers. Sometimes, such a sum can be added up to produce a finite answer, but this is not always the case. For example,

$$1 + 1 + 1 + 1 + 1 + \cdots = \infty. \tag{1.1}$$

Or, to illustrate even worse behaviour: What is the answer to

$$1 - 1 + 1 - 1 + 1 - 1 + \dots \quad ? \tag{1.2}$$

These series cannot be evaluated and are said to *diverge*. In order to introduce the notion of a *convergent* series, one that can be evaluated, let us begin by considering the following series:

$$S := \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}. \tag{1.3}$$

Each term is equal to one half of the previous term. Since we can deal with finite sums, let us consider

$$S_N := \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^N}. \tag{1.4}$$

Then we notice that

$$\frac{1}{2} S_N = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{N+1}}, \tag{1.5}$$

so that on subtracting (1.5) from (1.4), we see that

$$\frac{1}{2}S_N = \frac{1}{2} - \frac{1}{2^{N+1}} \tag{1.6}$$

and hence that

$$S_N = 1 - \frac{1}{2^N}. \tag{1.7}$$

Clearly S_N approaches 1 as N gets larger and larger. Indeed the series does sum to 1 according to the following well-established definition:

The series $\sum_{n=1}^{\infty} a_n$ is said to *converge* to L if for any $\epsilon > 0$, there exists a natural number N such that for every $n \geq N$ we have $|S_n - L| < \epsilon$. When this is the case we write $\sum_{n=1}^{\infty} a_n = L$.

Was there anything special about the value of 1/2 used here? A little more thought might lead us to consider the series

$$S = r + r^2 + r^3 + r^4 + r^5 + \cdots = \sum_{n=1}^{\infty} r^n. \tag{1.8}$$

A series like this, with a common ratio between consecutive terms is called a geometric series. If $r = 1$, then we have the series in (1.1). If $r = -1$, then we have the series in (1.2). So assume that $|r| \neq 1$. Now let us try to argue in exactly the same way as before: Defining $S_N = r + r^2 + \cdots + r^N$, we see that $rS_N = r^2 + r^3 + \ldots r^{N+1}$ and therefore that $(1-r)S_N = r - r^{N+1}$. Dividing through by $(1-r)$ we get that

$$S_N = \frac{r}{1-r} - \frac{r^{N+1}}{1-r}. \tag{1.9}$$

Now if $|r| < 1$, we can let $N \to \infty$ in the line above and deduce that

$$S = \frac{r}{1-r}. \tag{1.10}$$

If $|r| > 1$, then the series diverges. So there was nothing special about 1/2, but we needed to use a number smaller than 1 in modulus. Using the above formula, it is easy to work out, for example, that

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \frac{1}{2}. \tag{1.11}$$

2 Ratio and Root Tests

The series we considered in the previous section were those series $\sum_{n=1}^{\infty} a_n$ for which the equation

$$a_{n+1} = ra_n \tag{2.1}$$

held true for some fixed number r and for every $n \geq 1$. When $|r| < 1$ we saw that such series converged. What would happen if the above equation were only approximately true? Surely if it were *nearly* true, the series would still converge, *i.e.* a series $\sum_{n=1}^{\infty} a_n$ for which

$$a_{n+1} \approx ra_n \tag{2.2}$$

for some r with $|r| < 1$ would surely be convergent, provided we correctly interpret the '$\approx$' sign. Thinking along these lines leads us to the ratio test, the correct statement of which is:

Ratio Test. Suppose that the series $\sum_{n=1}^{\infty} a_n$ is such that

$$\left|\frac{a_{n+1}}{a_n}\right| \to r \quad \text{as} \quad n \to \infty. \tag{2.3}$$

If $r < 1$, then the series converges. If $r > 1$, then the series diverges. If $r = 1$, then the test is inconclusive.

For example, consider

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}. \tag{2.4}$$

The successive ratios here are

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{1}{2} + \frac{1}{2n}, \tag{2.5}$$

which tends to $1/2$ as $n \to \infty$. And so, by the ratio test, this series converges.

Another way to characterize those series that we considered in the previous section is those for which the equation

$$\sqrt[n]{|a_n|} = r \tag{2.6}$$

holds true for some fixed r and for every $n \geq 1$. Again, what if this were only approximately true?

Root Test. Suppose that the series $\sum_{n=1}^{\infty} a_n$ is such that

$$\sqrt[n]{|a_n|} \to r \quad \text{as} \quad n \to \infty. \tag{2.7}$$

If $r < 1$, then the series converges. If $r > 1$, then the series diverges. If $r = 1$, then the test is inconclusive.

For example, consider

$$\sum_{n=1}^{\infty} \left(\frac{n^2 + 2n + 3}{3n^2 + n - 1}\right)^n. \tag{2.8}$$

The n^{th} root of the n^{th} term is

$$\sqrt[n]{|a_n|} = \frac{n^2 + 2n + 3}{3n^2 + n - 1} = \frac{1 + 2/n + 3/n^2}{3 + 1/n - 1/n^2} \tag{2.9}$$

which tends to $1/3$ as $n \to \infty$. And so, by the root test, this series converges.

With these tests having been memorized, there is a sense in which any series the question of the convergence of which is reducible to one of these tests is less interesting, so we turn our attention to those series the convergence of which cannot be settled easily by the root or ratio tests.

3 The Harmonic Series

The following series is called the harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}. \tag{3.1}$$

Does it converge or diverge? Note that the ratio test is inconclusive, as is the root test (this is slightly harder to see but it amounts to the fact that $n^{1/n} \to 1$ as $n \to \infty$). Getting ones head around the divergence of the harmonic series is a classic issue one must face when learning basic analysis. The following proof of the divergence supposedly goes back to the 14th Century:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \sum_{k=1}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n} \\
&\geq 1 + \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^{k+1}} \\
&= 1 + \sum_{k=1}^{\infty} \frac{1}{2} \\
&= \infty.
\end{aligned}$$

Thinking more about exactly why this proof works leads to the *Cauchy Condensation Test*, which we will not discuss in detail here.

Remark 3.1. *The partial sums*

$$H_N = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \tag{3.2}$$

are called the harmonic numbers. It can be shown that $H_N \approx \log N$*. To give you some idea about just how slowly this sum diverges, observe that* $\log 10^{30} < 100$ *(so after summing* 10^{30} *terms of the series, the total is less than 100). Jeffrey Lagarias proved in 2001 that the conjecture*

$$\sum_{d|n} d \leq H_n + e^{H_n} \log H_n \tag{3.3}$$

for all $n \geq 1$ *is equivalent to the Riemann Hypothesis, which is one of the great unsolved conjectures in number theory.*

A more modern trick to show the divergence of the harmonic series would be to compare it to a divergent integral:

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \sum_{n=1}^{\infty} \int_{n}^{n+1} \frac{1}{x}\, dx = \int_{1}^{\infty} \frac{1}{x}\, dx = \infty. \tag{3.4}$$

Similarly, the convergence of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{3.5}$$

can be established by comparison with an integral:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{1}{x^2} dx = 1 + \int_{1}^{\infty} \frac{1}{x^2} dx = 2. \tag{3.6}$$

Thinking more along these lines leads to the following test:

Integral Test. Let $f : [0, \infty) \to [0, \infty)$ be a monotone decreasing function. For any number N, the series $\sum_{n=N}^{\infty} f(n)$ converges if and only if the corresponding integral $\int_N^\infty f(x)dx$ is finite.

Remark 3.2. *Since the integral $\int_1^\infty \frac{1}{x^{1+\epsilon}}dx$ is finite for any $\epsilon > 0$, the integral test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}}$ converges for every $\epsilon > 0$ but diverges for $\epsilon = 0$.*

Remark 3.3. *One can also consider the alternating harmonic series:*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}. \tag{3.7}$$

This series converges to $\log 2$ but has the remarkable property that given any real number α, the terms can be rearranged into a series that converges to α.

4 Large Subsets of Natural Numbers

The first geometric series we considered, the harmonic series and the sums of the reciprocals of the square numbers are all series of the following form: Given an infinite subset $A \subset \mathbb{N}$, we define

$$S[A] := \sum_{n \in A} \frac{1}{n}. \tag{4.1}$$

Based on whether or not this series converges, we can divide the set of infinite subsets of $\mathbb{N}$ into two classes. We will call a subset $A \subset \mathbb{N}$ *large* if $S[A] = \infty$. We showed in the previous section that $\mathbb{N}$ itself is large, but that the set of square numbers is not large.

For example, Start with $\mathbb{N}$ and remove any natural numbers whose decimal expansion contains a 7. Call the remaining set A_7. Is A_7 large? On the one hand we are simply in a situation where we can use 9 digits instead of 10 digits to build numbers from, so surely nothing has really changed and A_7 remains large. On the other hand, it seems we *have* removed a lot of numbers since for a large number it seems almost overwhelmingly likely that it contains a 7 somewhere, so this heuristic suggest A_7 is not large. This latter intuition is correct. We split the sum up using the number of digits of n; observe that k-digit numbers n are precisely those satisfying $10^{k-1} \leq n \leq 10^k - 1$, so

$$\sum_{n \in A_7} \frac{1}{n} = \sum_{k=1}^{\infty} \sum_{\substack{n \in A_7 \\ n \text{ has } k \text{ digits}}} \frac{1}{n} = \sum_{k=1}^{\infty} \sum_{\substack{n \in A_7 \\ 10^{k-1} \leq n \leq 10^k - 1}} \frac{1}{n} \tag{4.2}$$

Now, the number of k-digit numbers that belong to A_7 is exactly

$$8 \cdot 9^{k-1}, \tag{4.3}$$

because there are 8 choices ($\{1,2,3,4,5,6,8,9\}$) for the first digit and 9 choices for each of the remaining $(k-1)$ digits. And, if n is such a k-digit number, then

$$\frac{1}{n} \le \frac{1}{10^{k-1}}. \tag{4.4}$$

So,

$$\begin{aligned}
&\sum_{k=1}^{\infty} \sum_{\substack{n \in A_7 \\ 10^{k-1} \le n \le 10^k - 1}} \frac{1}{n} \\
&\le \sum_{k=1}^{\infty} (\text{no. of } k \text{ digit numbers in } A_7) \times \frac{1}{10^{k-1}} \\
&\le \sum_{k=1}^{\infty} \frac{8 \cdot 9^{k-1}}{10^{k-1}} \\
&= 8 \sum_{k=1}^{\infty} \left(\frac{9}{10}\right)^{k-1} \\
&< \infty,
\end{aligned}$$

where we have summed a geometric series in the last step. So A_7 is not large.

4.1 The Set of Prime Numbers is Large

The set of prime numbers is large, which is somewhat surprising because we expect primes to become rare quite quickly as numbers get very large. Compare this with the fact that the set of perfect squares is not large and yet one can always predict where the next perfect square is (it's only $(n+1)^2 - n^2 = 2n+1$ numbers away!).

To prove that the set of primes is large, we assume for the sake of contradiction that it is not large: This means that there is some prime, the k^{th} prime p_k, say, which is such that the sum over all primes larger than p_k is less than 1/2, *i.e.*

$$\sum_{p > p_k} \frac{1}{p} < 1/2. \tag{4.5}$$

Now we fix an $N \ge 1$ and ask the question: How many numbers between 1 and N are divisible by some prime p that is bigger than p_k? Well, if a number is divisible by p_{k+1}, then it is a multiple of p_{k+1} and there at at most N/p_{k+1} such numbers between 1 and N. Similarly, at most N/p_{k+2} such numbers are divisible by p_{k+2} and so forth. Therefore at most

$$\frac{N}{p_{k+1}} + \frac{N}{p_{k+2}} + \cdots = N \sum_{p > p_k} \frac{1}{p} < N/2 \tag{4.6}$$

of the numbers between 1 and N are divisible by some prime that is bigger than p_k. The rest (*i.e.* at least $N/2$) of the numbers between 1 and N are divisible *only* by the primes that are less than or equal to p_k. Let us estimate this independently from above: So we ask how many numbers between 1 and N are divisible only by the primes that are less than or equal to p_k? Well for each such number n, write it as

$$n = sm^2. \tag{4.7}$$

where s is a square-free integer [1]. Since n is a product of powers of p_1,...,p_k, there are at most 2^k possible choices for s. Then, since $n \leq N$, we have that $m^2 \leq N$, and thus there are at most $\sqrt{N}$ possible choices for m. Thus we have that

$$N/2 \leq 2^k \sqrt{N} \tag{4.8}$$

for some fixed k and for every N. This is of course false and this contradiction proves that the set of primes is large.

Remark 4.1. ***1.*** *Erdos conjectured that every large set contains arbitrarily long arithmetic progressions. There has been little progress on this conjecture. In fact, it is now known whether or not a large set must contain a single 3-term arithmetic progression. The best possible result in this direction is implied by the work of Tom Sanders ([San11]). It was however proved by Ben Green and Terence Tao in 2004 that the primes contain arbitrarily long arithmetic progressions ([GT08]).*

2. *The Twin Prime conjecture states that* $\liminf_n(p_{n+1} - p_n) = 2$ *(*i.e. *there are infinitely many pairs of primes of the form p, p+2). It is still open but it was shown in 2013 by Yitang Zhang ([Zha13]) that*

$$\liminf_n (p_{n+1} - p_n) \leq C. \tag{4.9}$$

Zhang got $C = 70,000,000$*, but since then the bound has been improved by James Maynard, Terence Tao and a Polymath project. It is claimed that the current 'World Record' is*

$$\liminf_n (p_{n+1} - p_n) \leq 246. \tag{4.10}$$

To relate this to our notion of large sets, consider that Brun proved in 1919 that the set of Twin primes is small (if it were large it would of course imply the twin prime conjecture, which is still open. In fact, if the limit of this sum, Brun's constant, were irrational it would imply the twin prime conjecture).

References

[GT08] Ben Green and Terence Tao, *The primes contain arbitrarily long arithmetic progressions*, Annals of Mathematics (2008), 481–547.

[San11] Tom Sanders, *On roths theorem on progressions*, Annals of Mathematics **174** (2011), no. 1, 619–636.

[Zha13] Yitang Zhang, *Bounded gaps between primes*, Annals of Mathematics (2013).

[1] This means that if $n = p_1^{a_1} \cdot \dots \cdot p_k^{a_k}$, then for each a_i we write $a_i = 2b_i + \epsilon_i$, where $\epsilon_i \in \{0,1\}$ and set $s = p_1^{\epsilon_1} \cdot \dots \cdot p_k^{\epsilon_k}$ and $m = p_1^{b_1} \cdot \dots \cdot p_k^{b_k}$