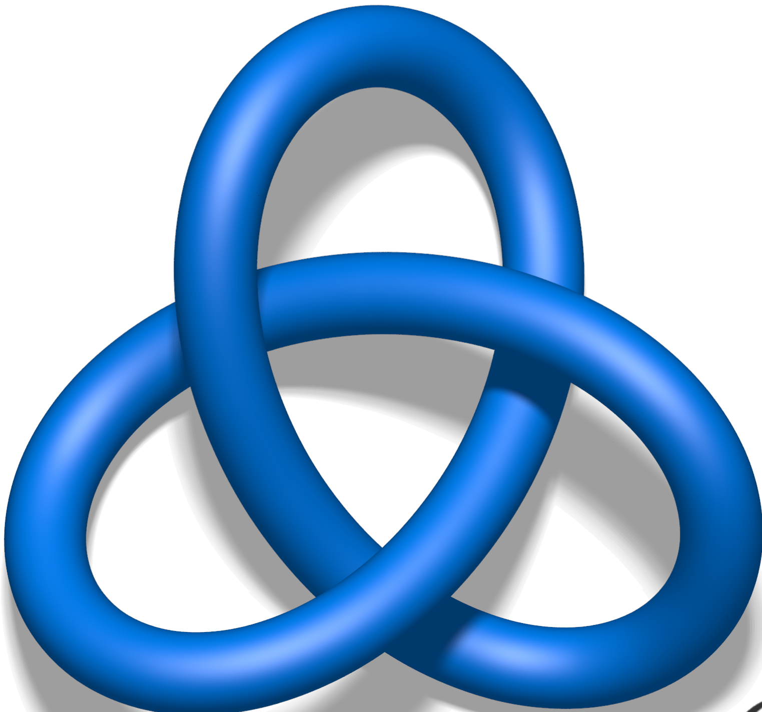
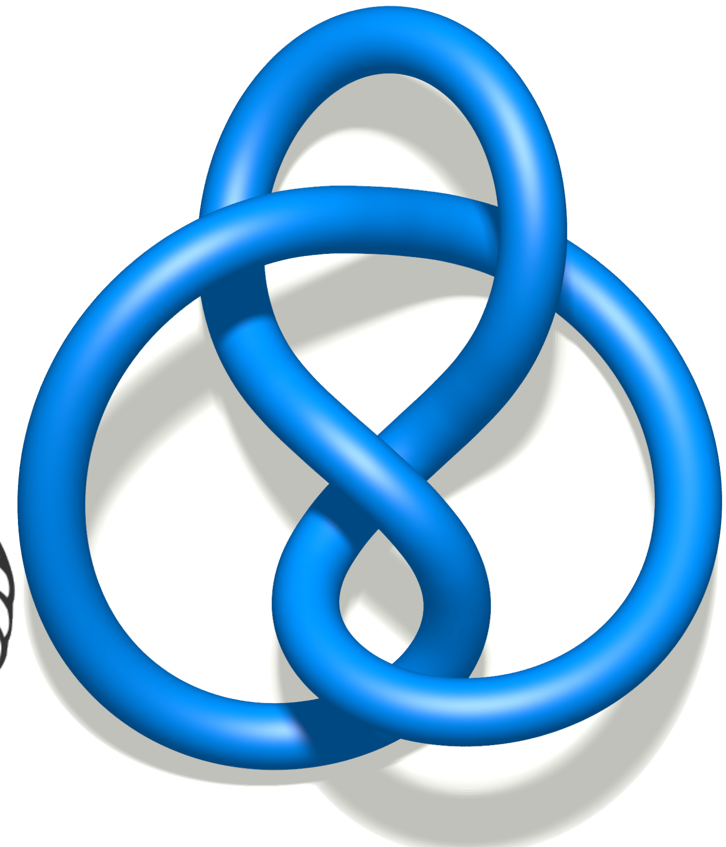

Knots and how to
detect knotting.



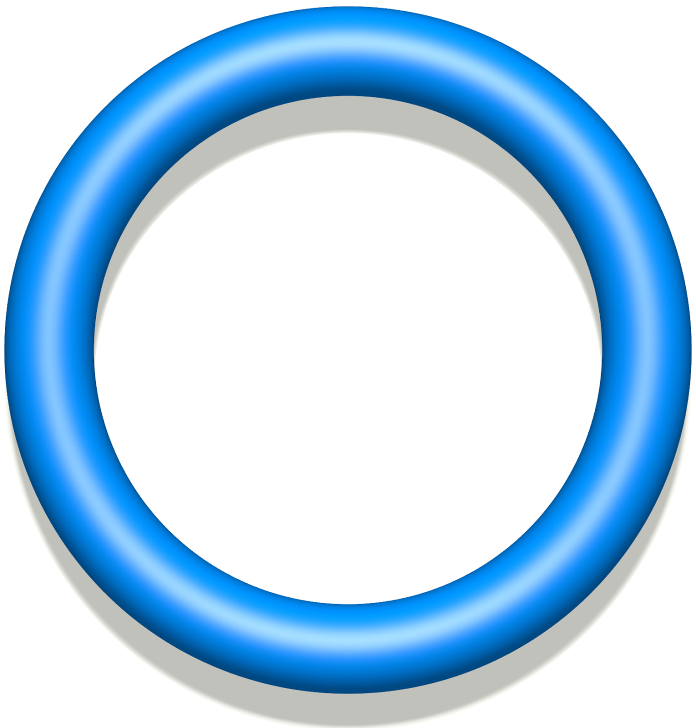
Trefoil knot

Figure-8 knot

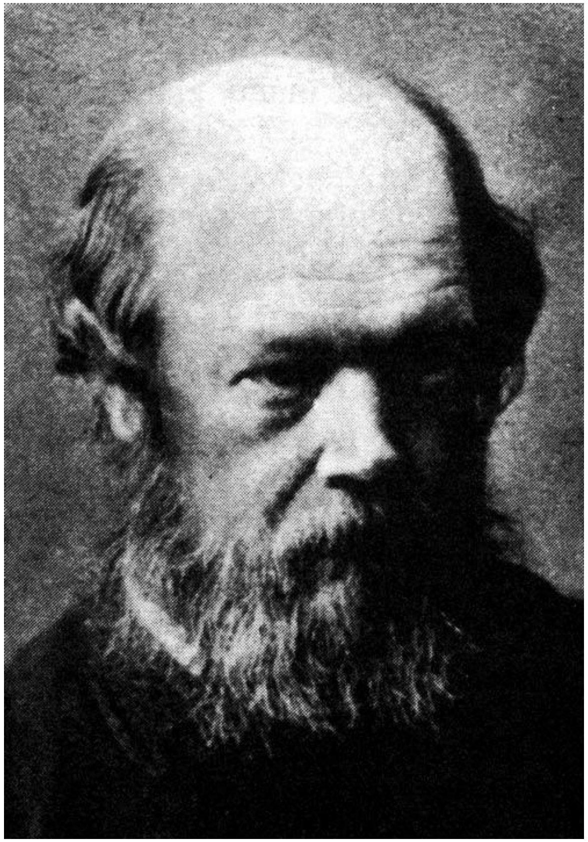


The “unknot”

(a mathematician’s “joke”)

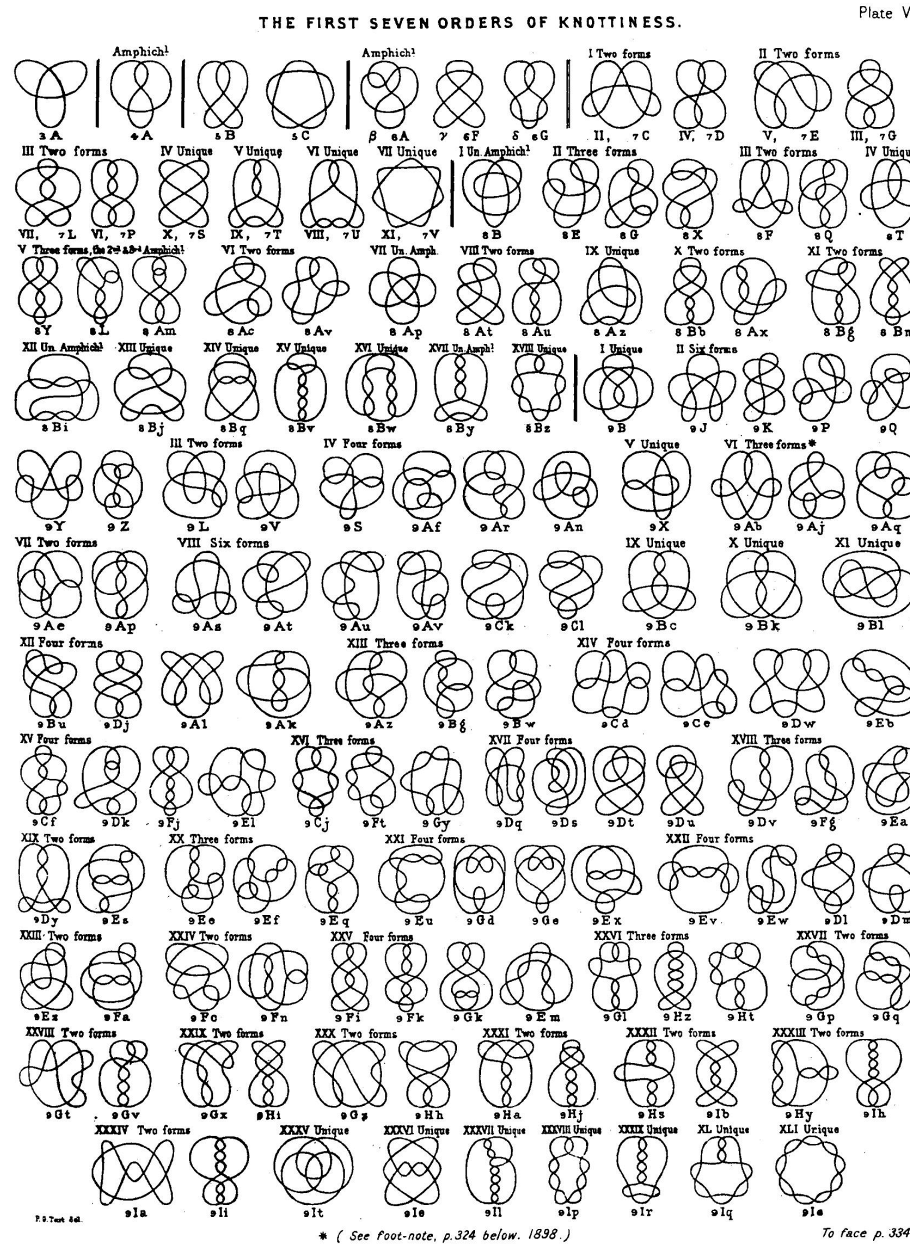


There are lots and lots of knots …

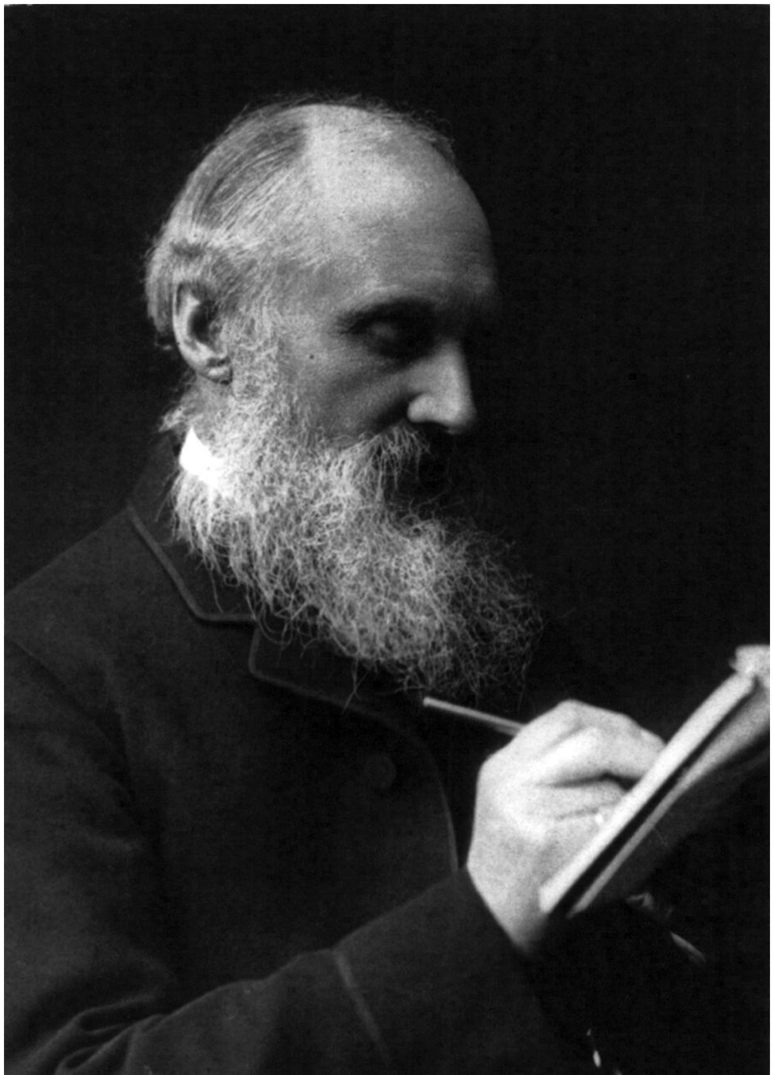


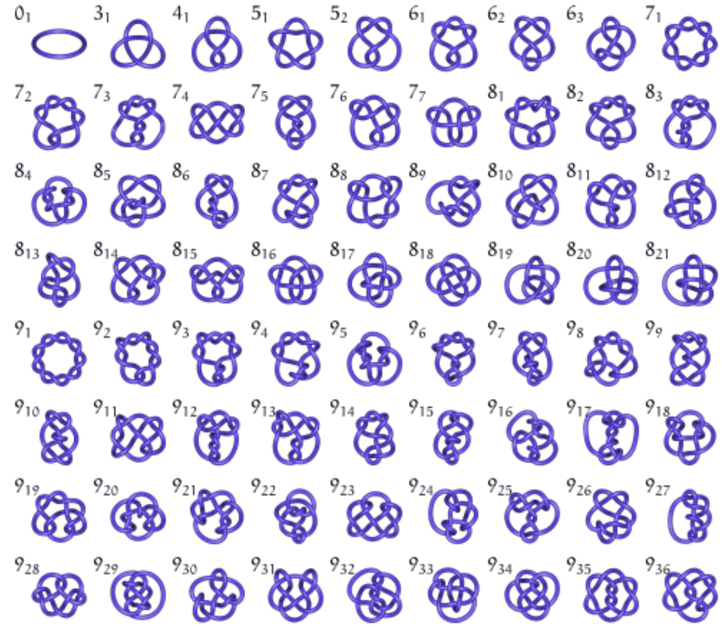
Peter Guthrie Tait

Tait's dates: 1831-1901



Lord Kelvin
(William Thomson)



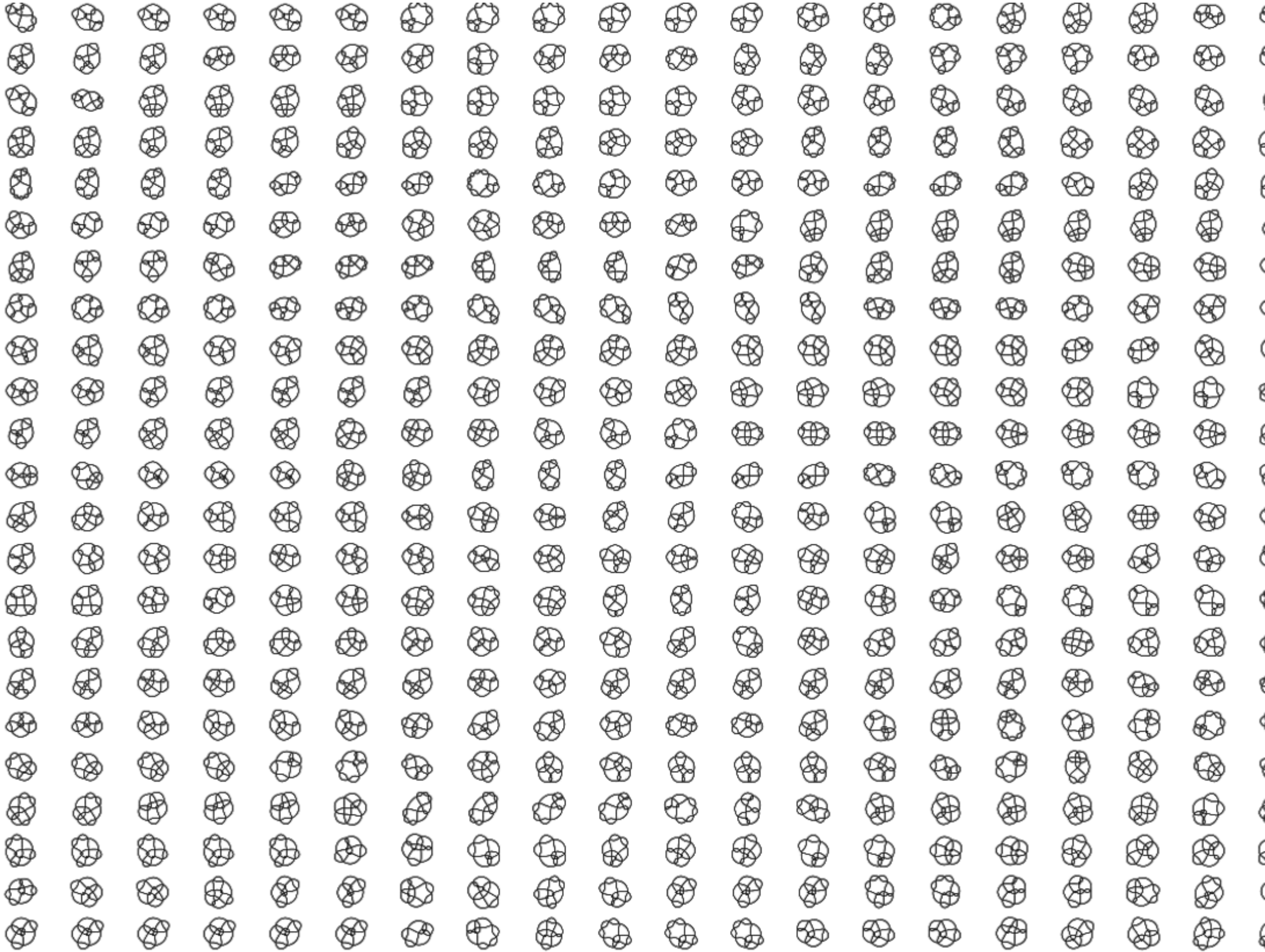


?

1 H																	2 He
3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
55 Cs	56 Ba	57 -71	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
87 Fr	88 Ra	89 -103	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Uut	114 Fl	115 Uup	116 Lv	117 Uus	118 Uuo

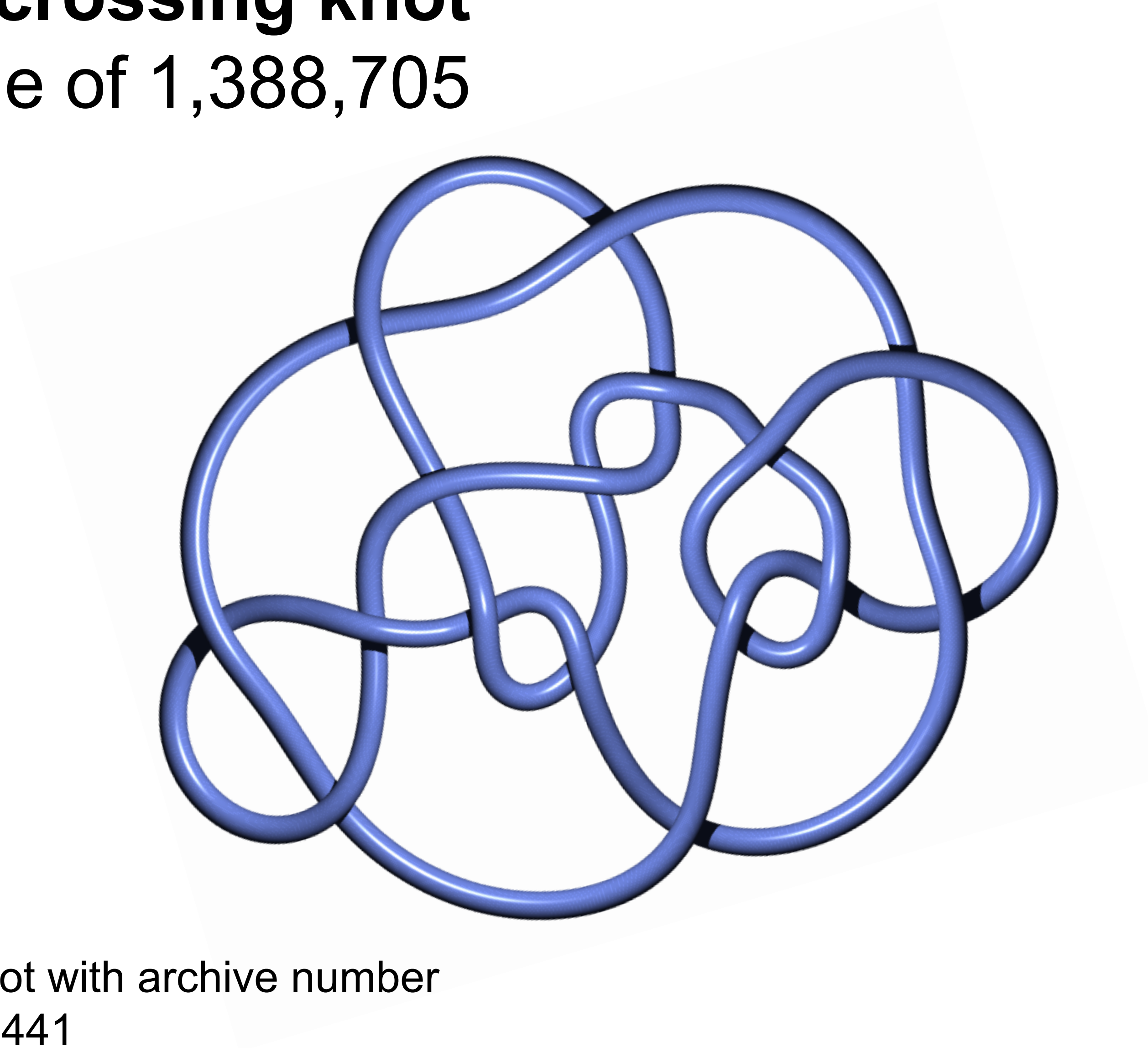
57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu
89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr

Known in antiquity
also known when(akw) Levoisier published his list of elements(1789)
akw Mendeleev published his periodic table(1869)
akw Deming published his periodic table(1923)
akw Seaborg published his periodic table(1945)
also known(ak) up to 2000
ak to 2012



A 16-crossing knot

one of 1,388,705

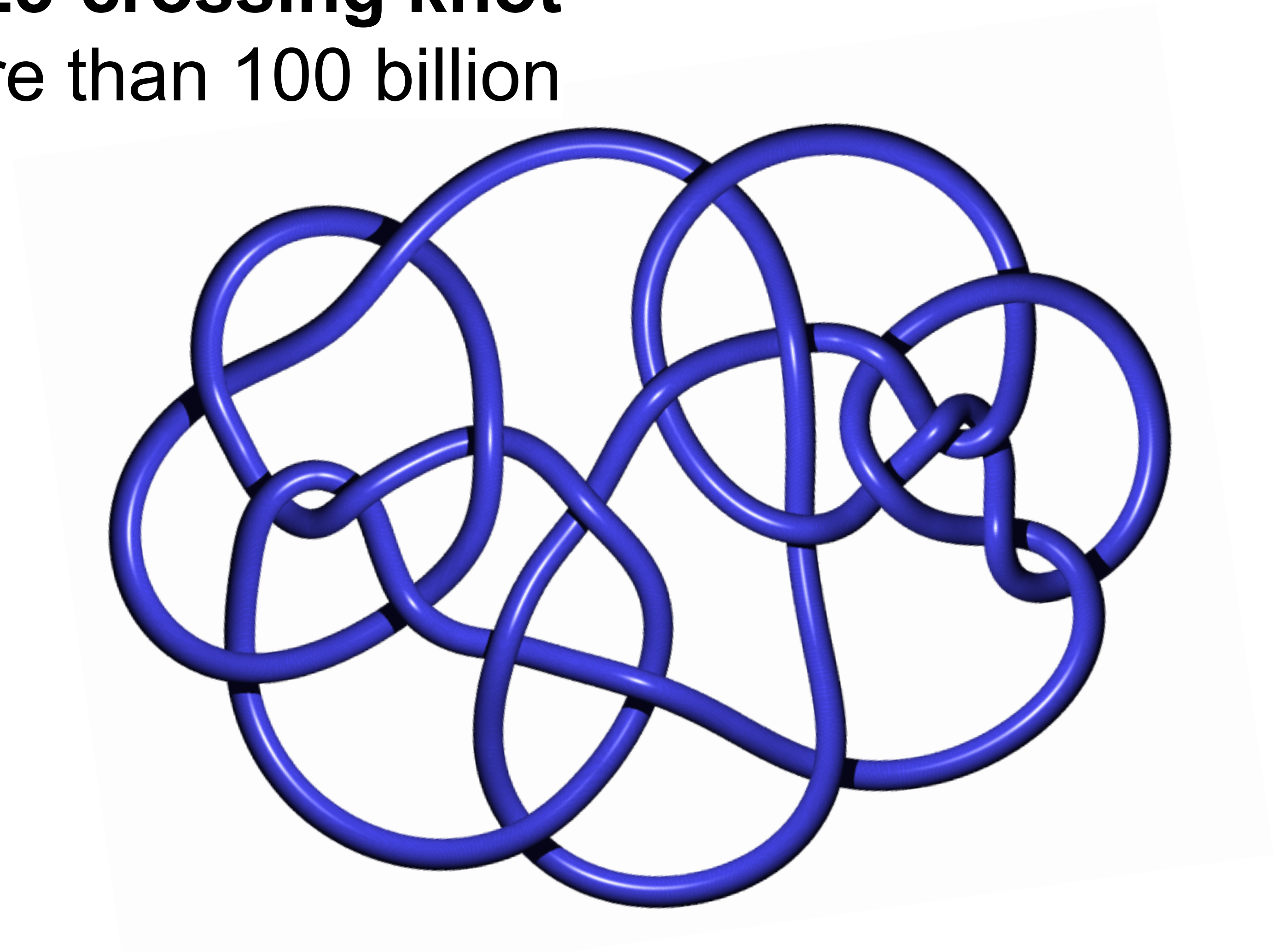


The knot with archive number 16n-63441

Image generated at http://knotilus.math.uwo.ca/

A 23-crossing knot

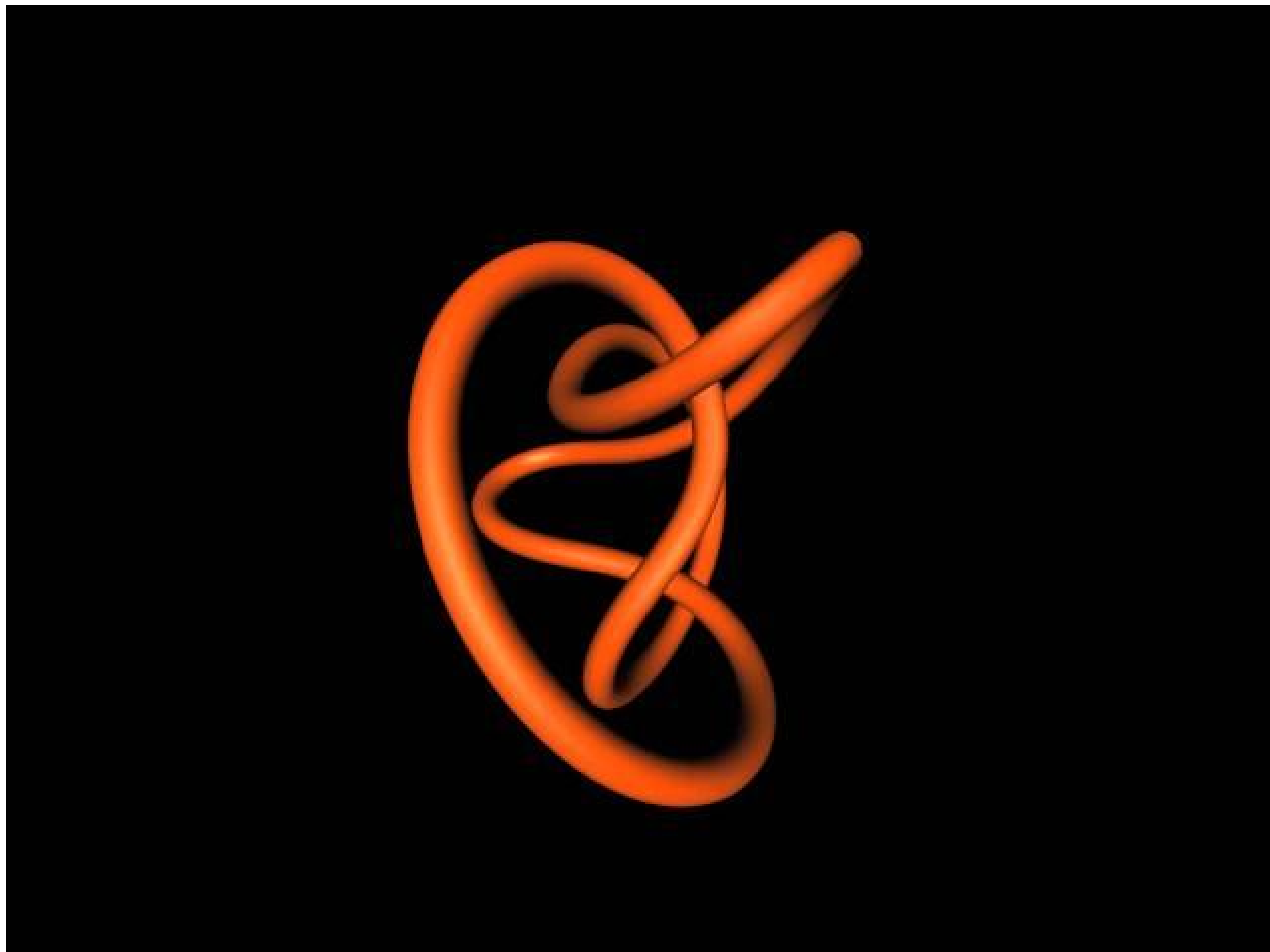
one of more than 100 billion



The knot with archive number
23x-1-25182457376

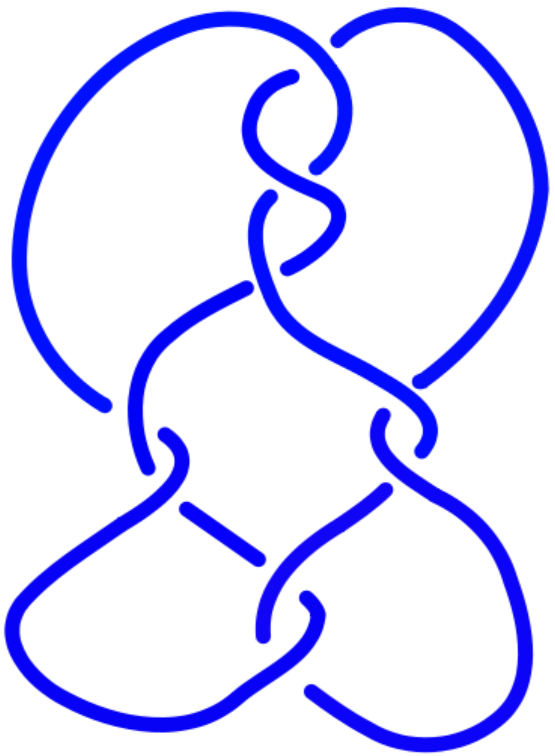
Image generated at http://knotilus.math.uwo.ca/

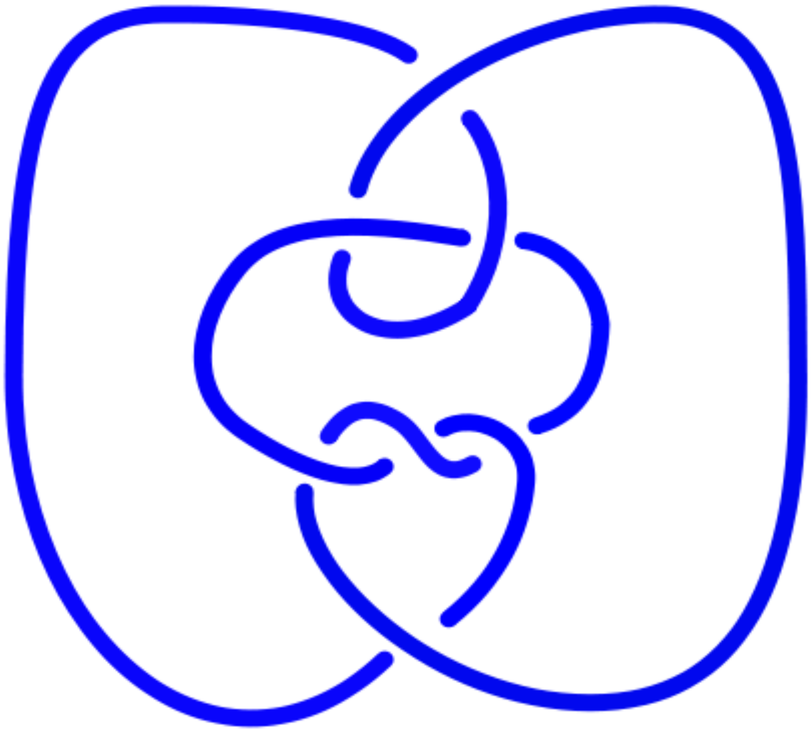
Spot the knot



Video: Robert Scharein
knotplot.com

Some other hard unknots

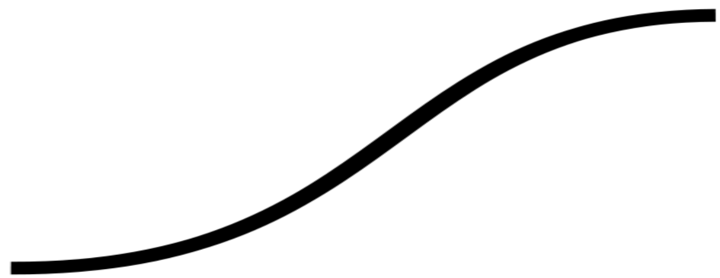




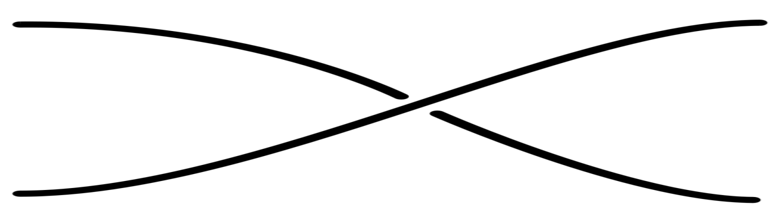
Knot Projections

A typical way of encoding a knot is via **projection**. This of the knot **K** sitting in 3-space with coordinates (x,y,z) and project to the xy-plane. We remember the image of the projection together with the over and under crossing information as in some of the picture we drawn.

Any “generic” picture of a knot only has two picture near a given point on the knot. Either a nice smooth arc.

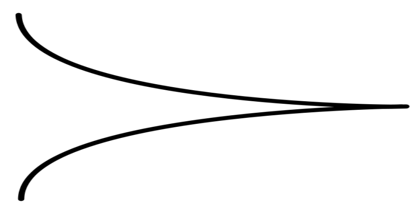


Or two smooth curve crossing over each each other with distinct tangent lines.

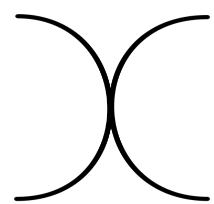


In particular we don’t have:

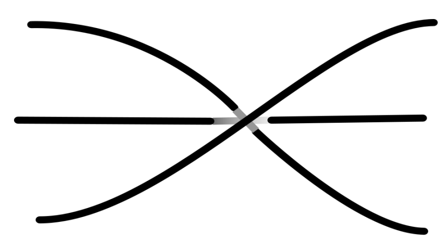
Cusps (where the knot has vertical tangent.)



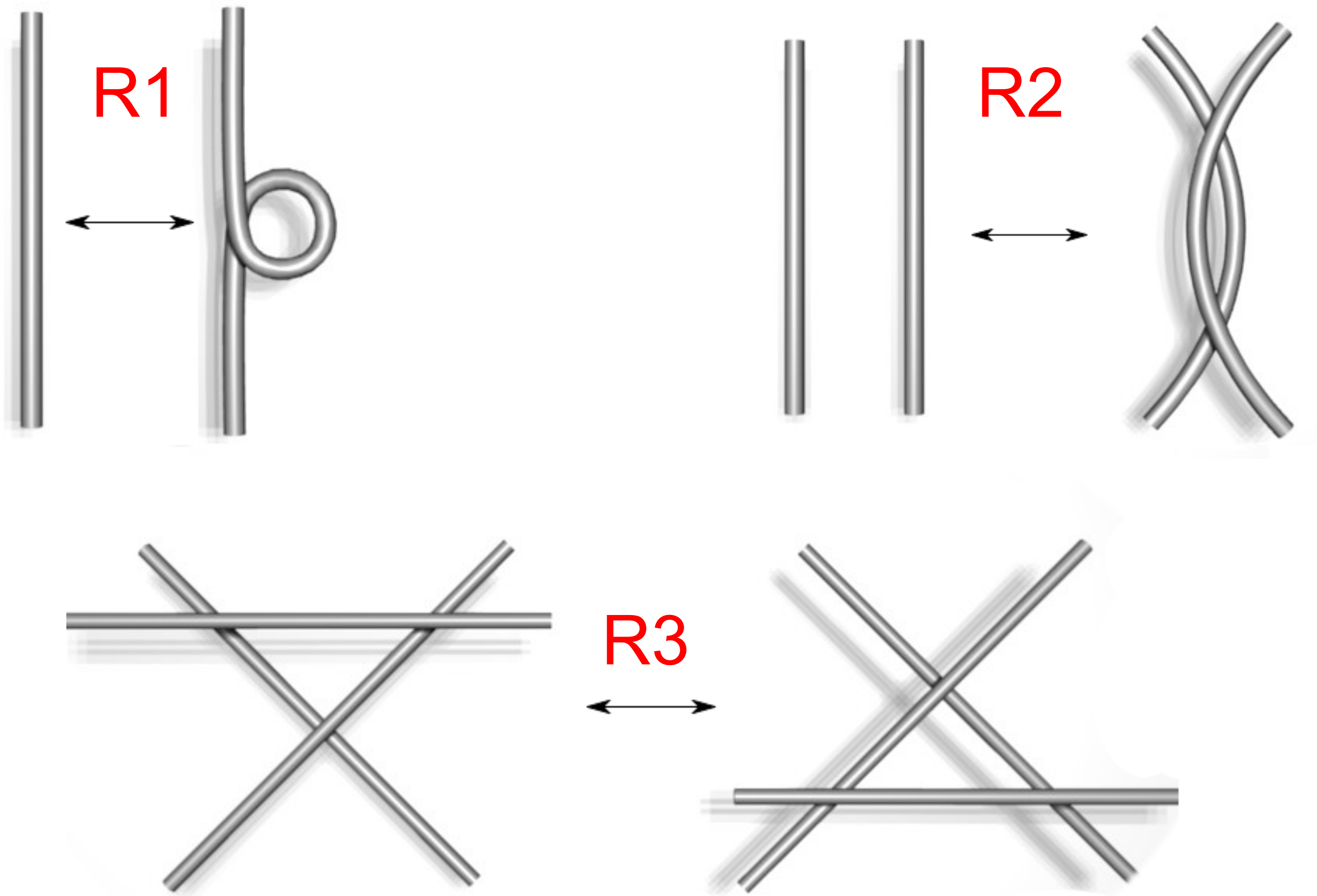
Common tangents.



Or triple points.



Reidermeister Moves:



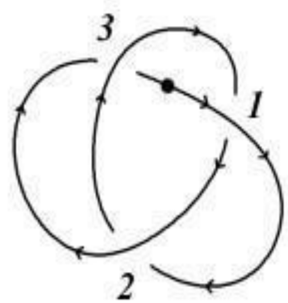
Other ways of specifying a knot

Gauss Code or Gauss Diagram: Label the crossings (say by 1,2,...n).

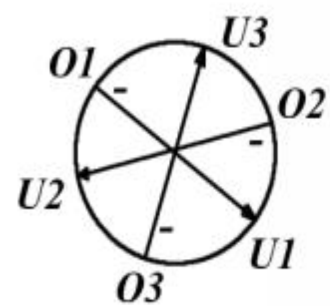
1. Pick a base point on the knot and an orientation.
2. Run along the knot and write down the label of the crossing you meeting together with a O/U if you are going over or under.

To get the Gauss diagram

1. Record the code on a circle.
2. Join points labelled by the same number by directed edge going from the over crossing to the undercrossing.
3. Give the edges a sign according to whether the crossing is positive or negative.



O1-U2-O3-U1-O2-U3-

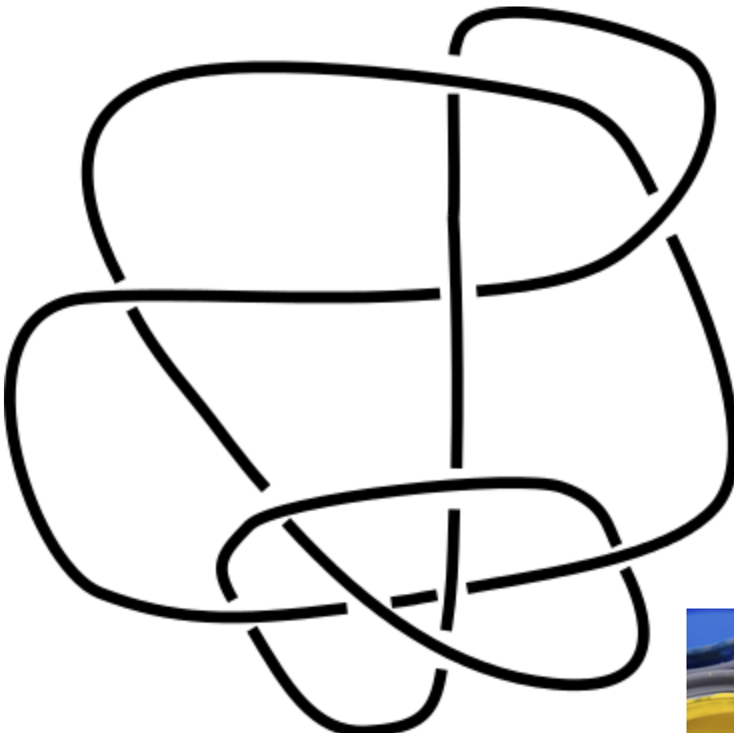


Gauss Diagram

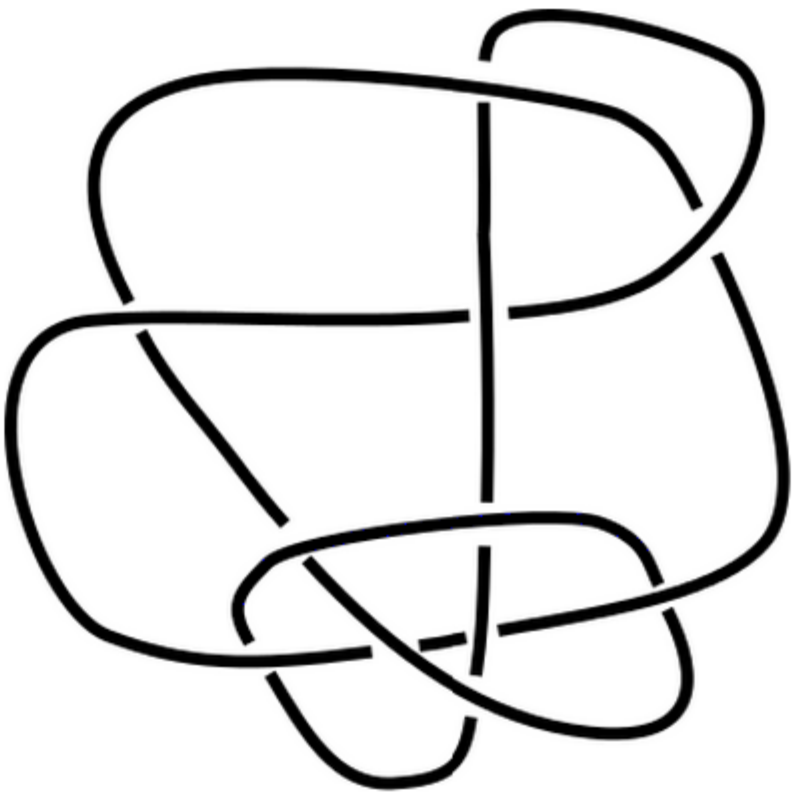
Signed planar graphs: Black and white color the knot complement.

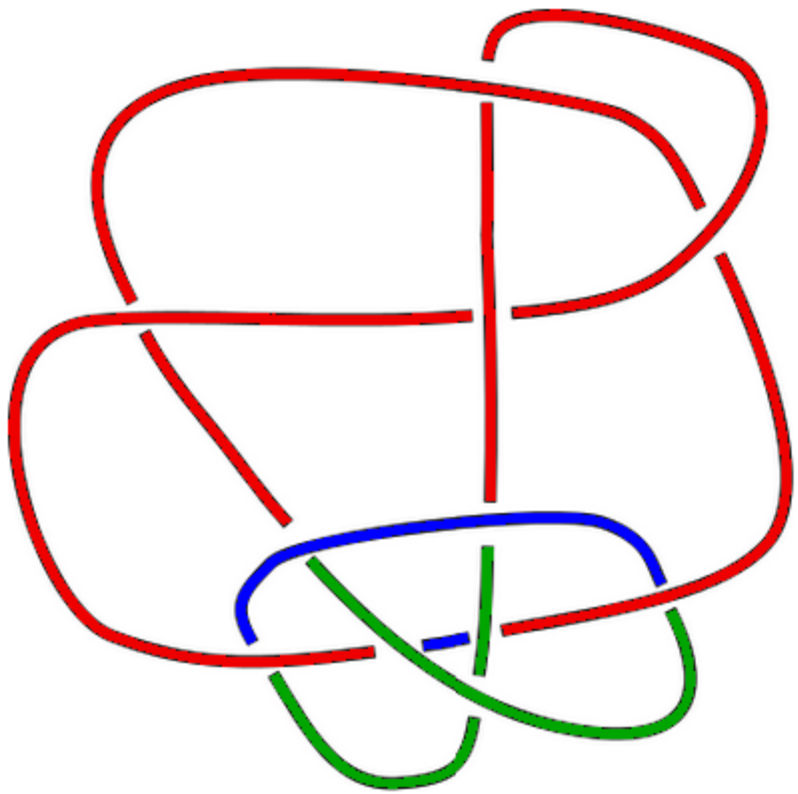
1. Make a graph whose vertices are the black regions.
2. draw an edge between vertices if the corresponding regions touch.
3. Label the edges with a +- according to the handedness of the twisting when the black regions meet.

How do we know it is knotted?

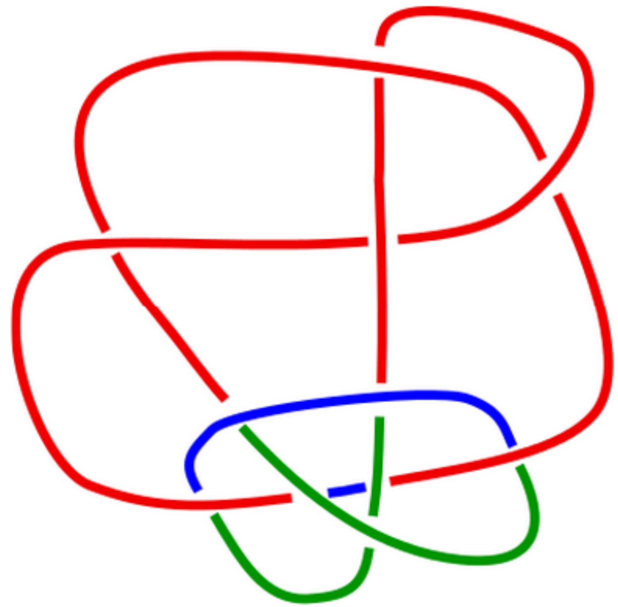


One way is by coloring it …

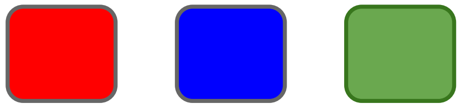




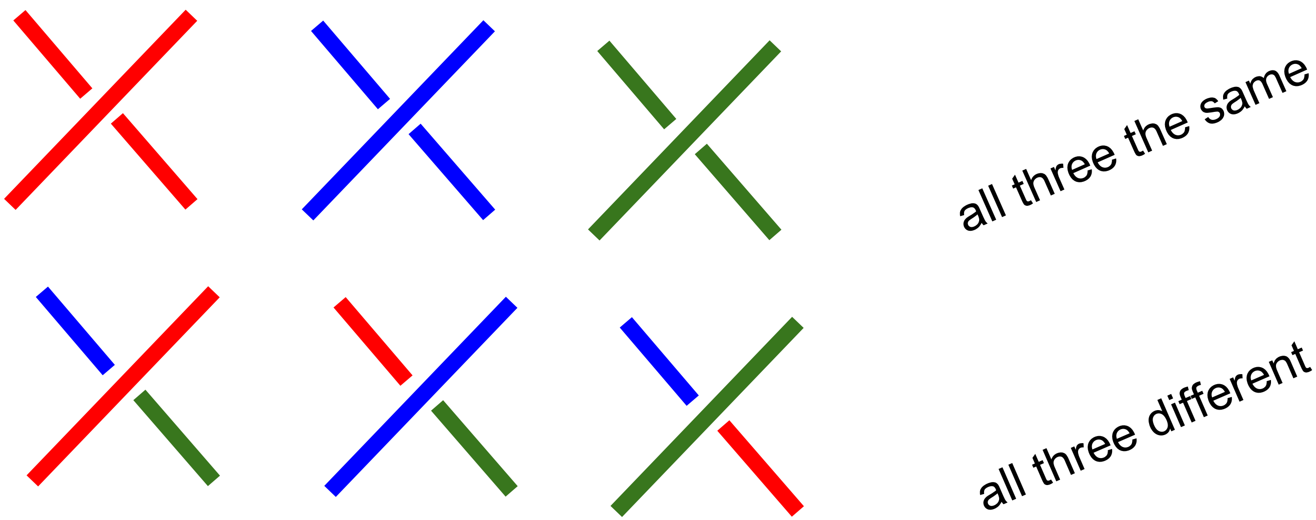
Rules of the coloring game:

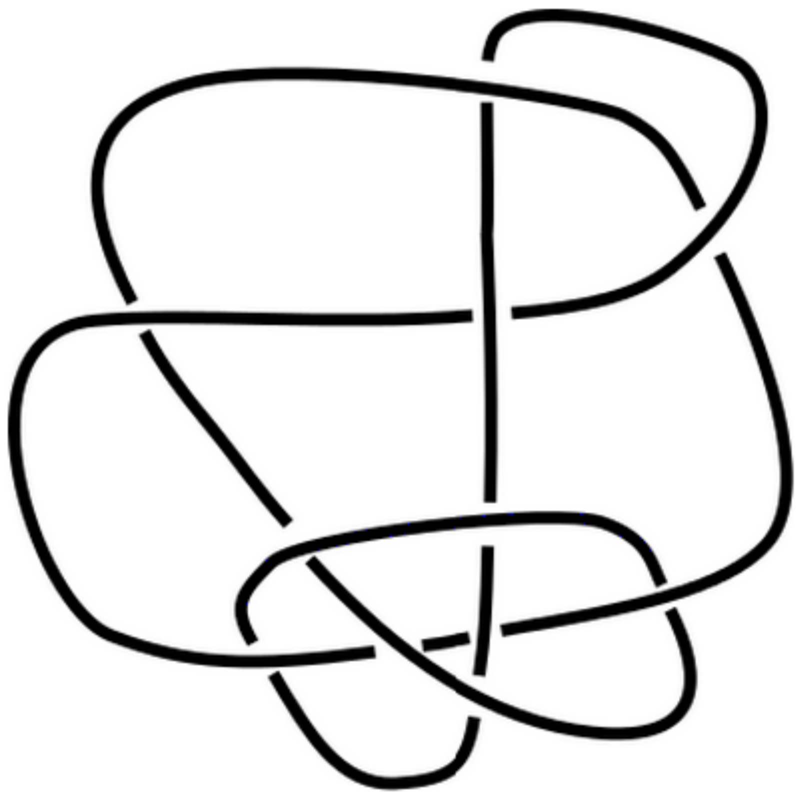


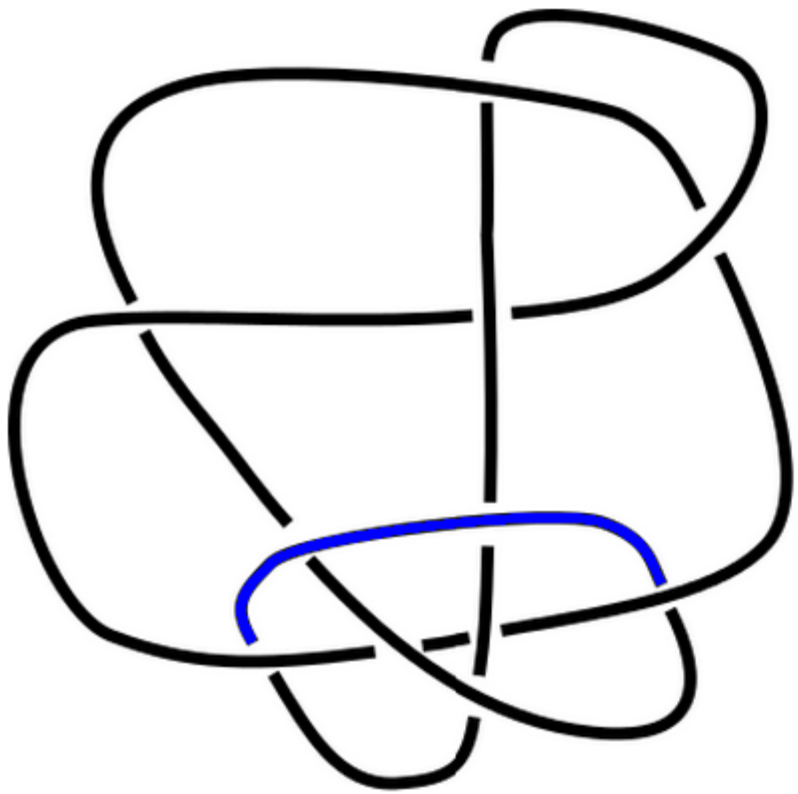
- only three colors (for now)

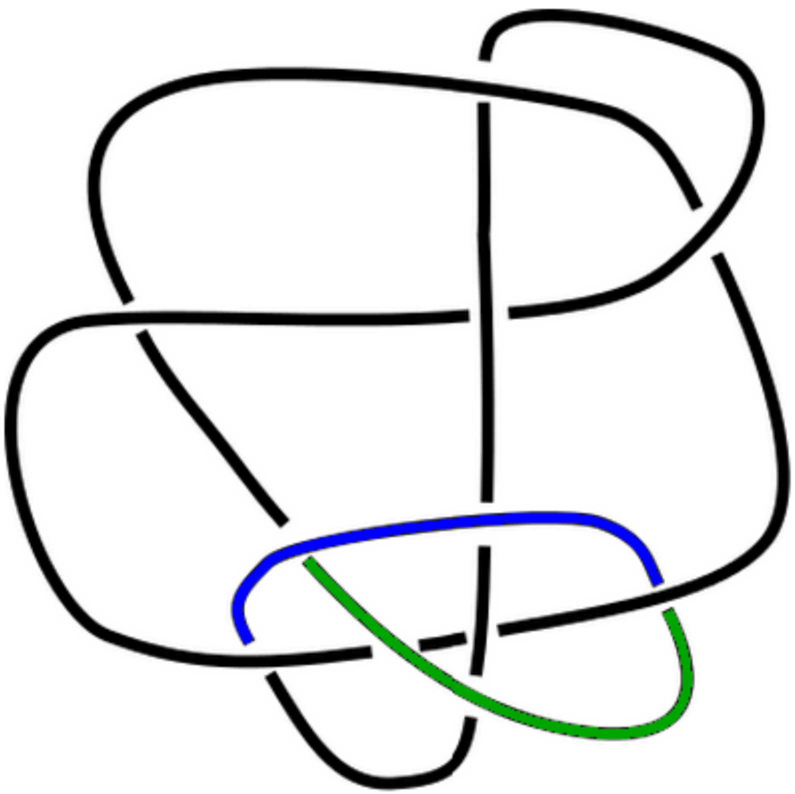


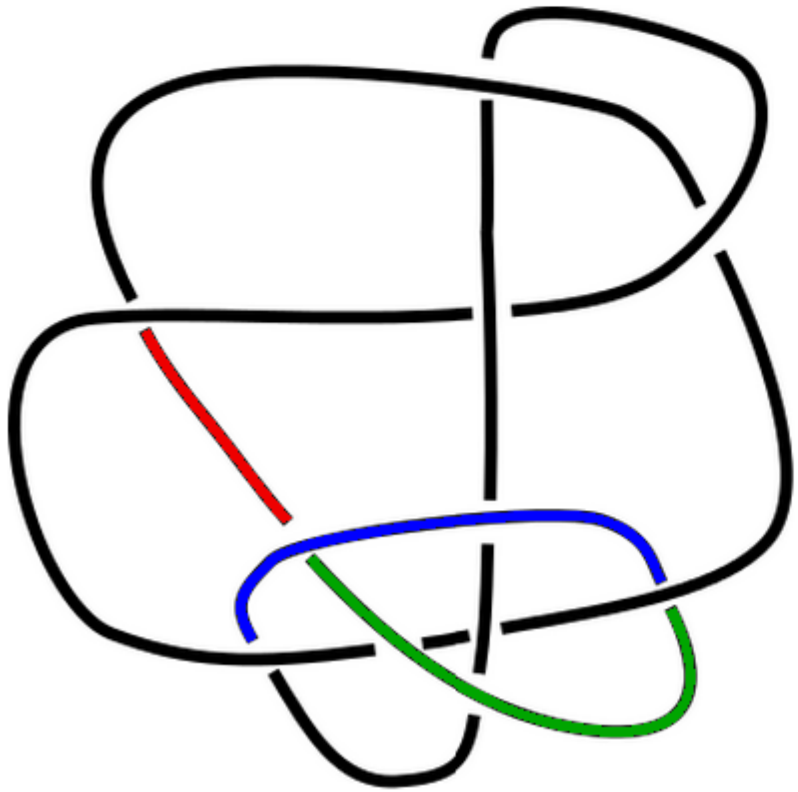
- at each crossing, the pattern is one of these:

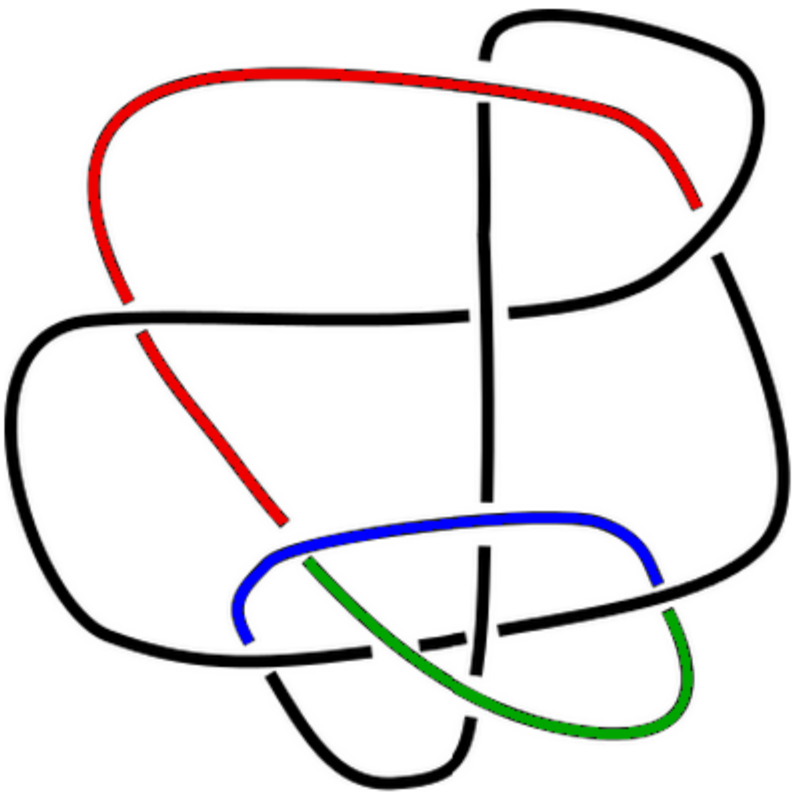


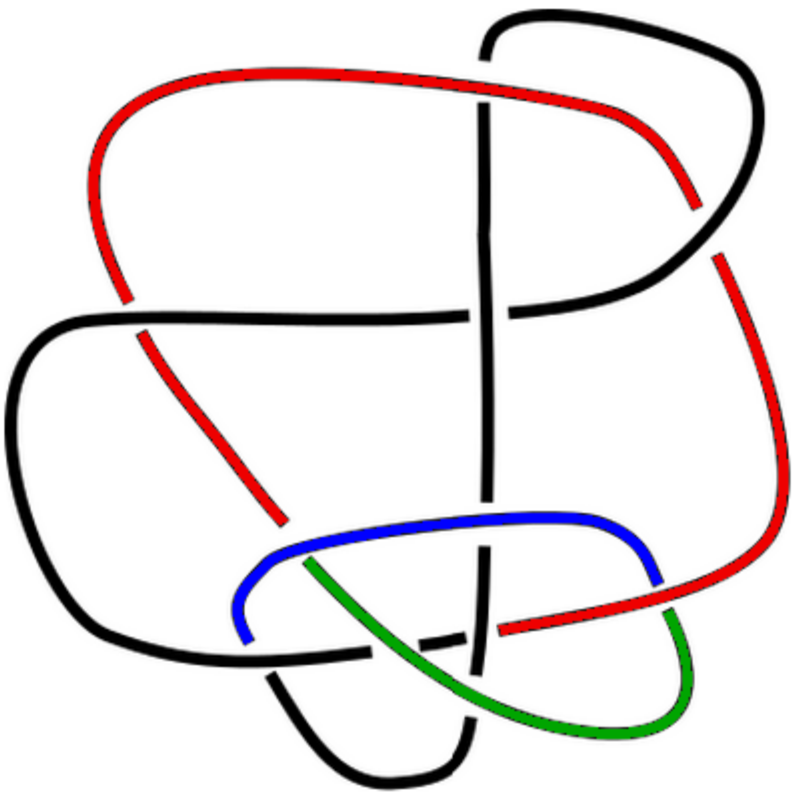


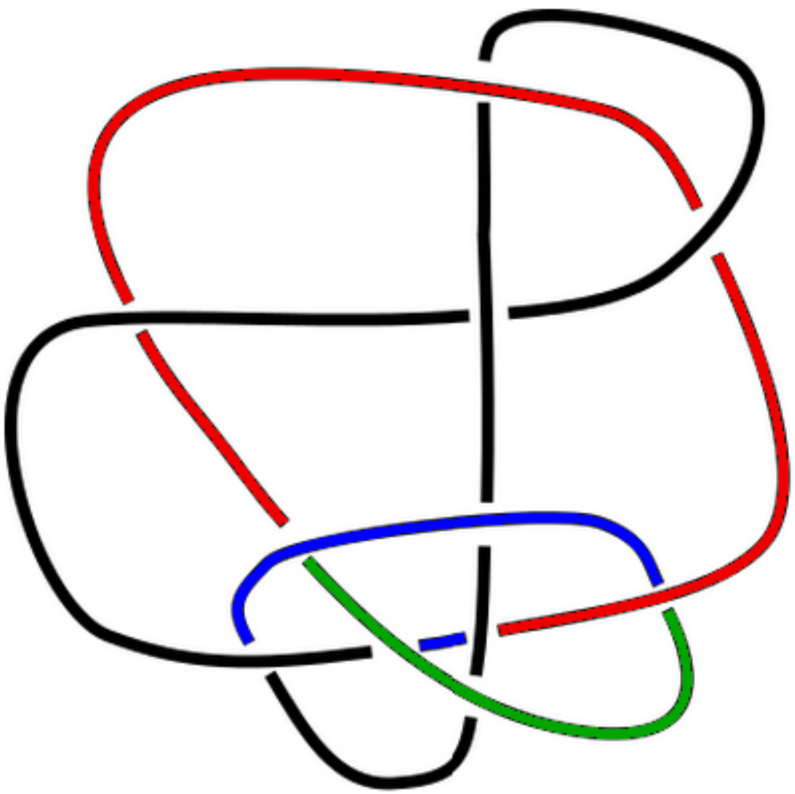


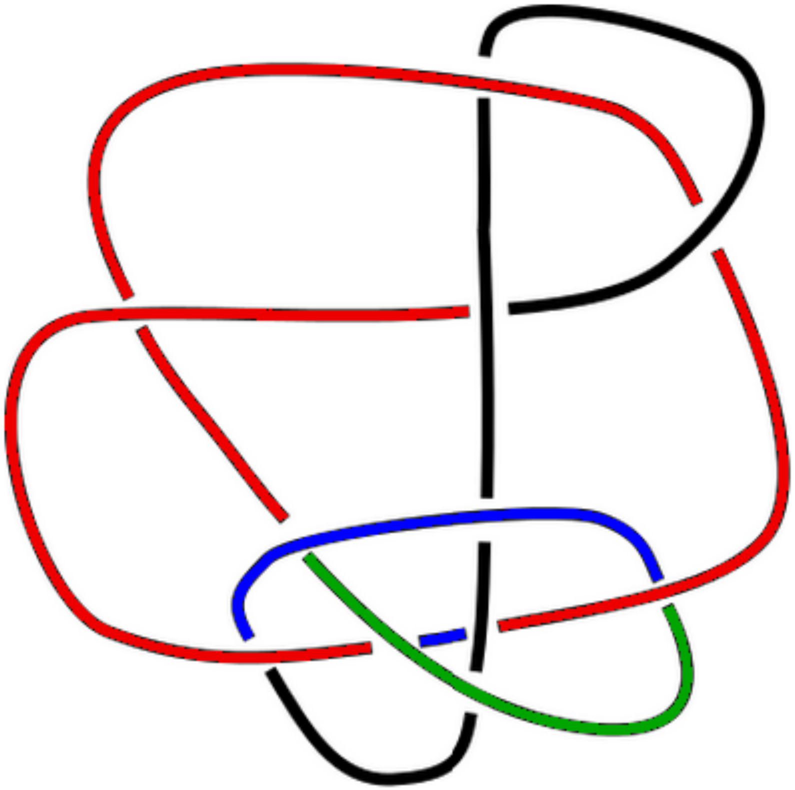


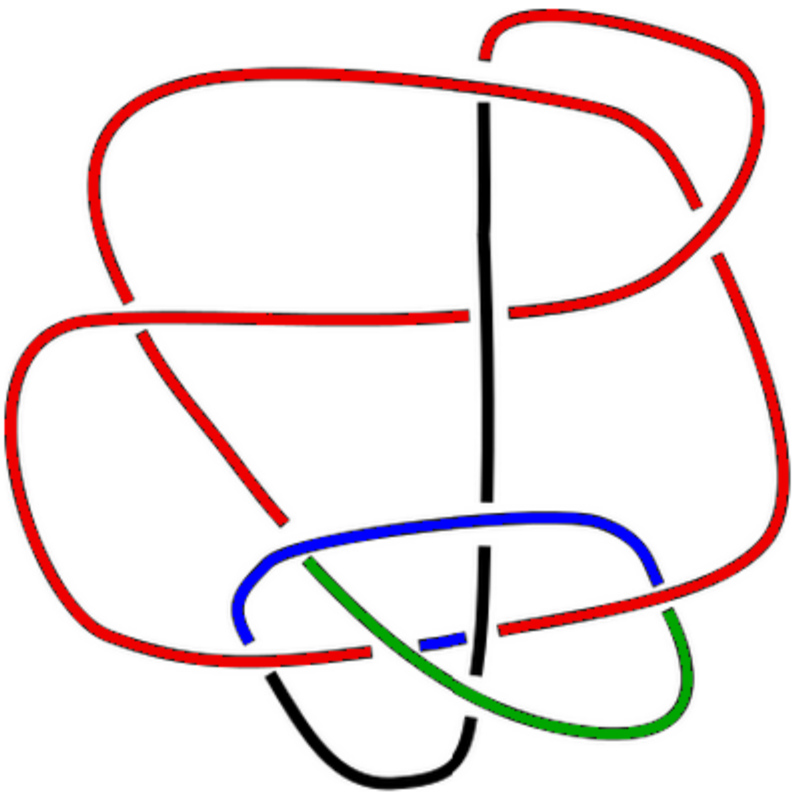


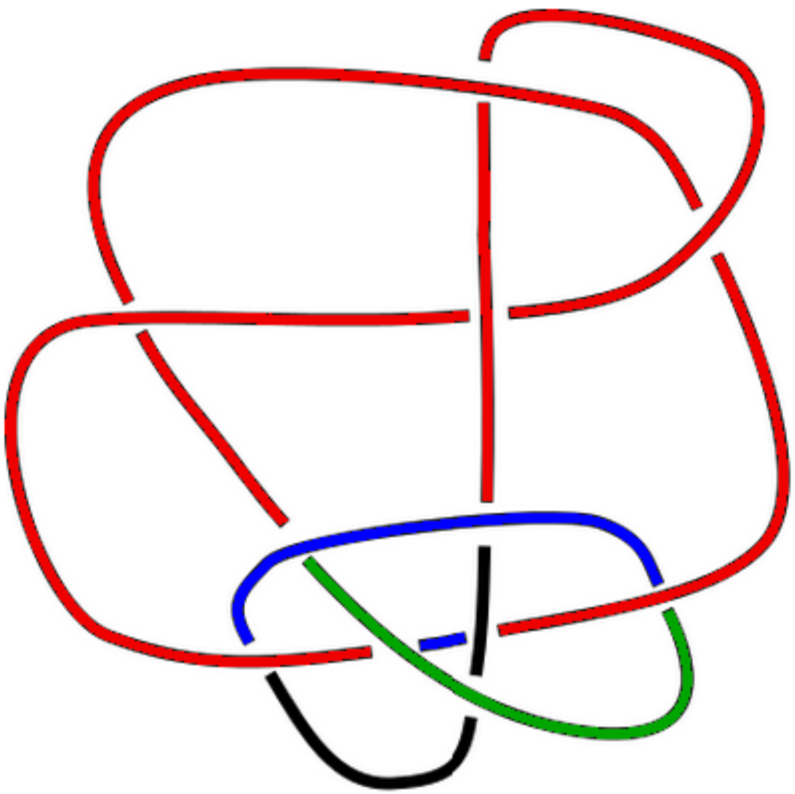


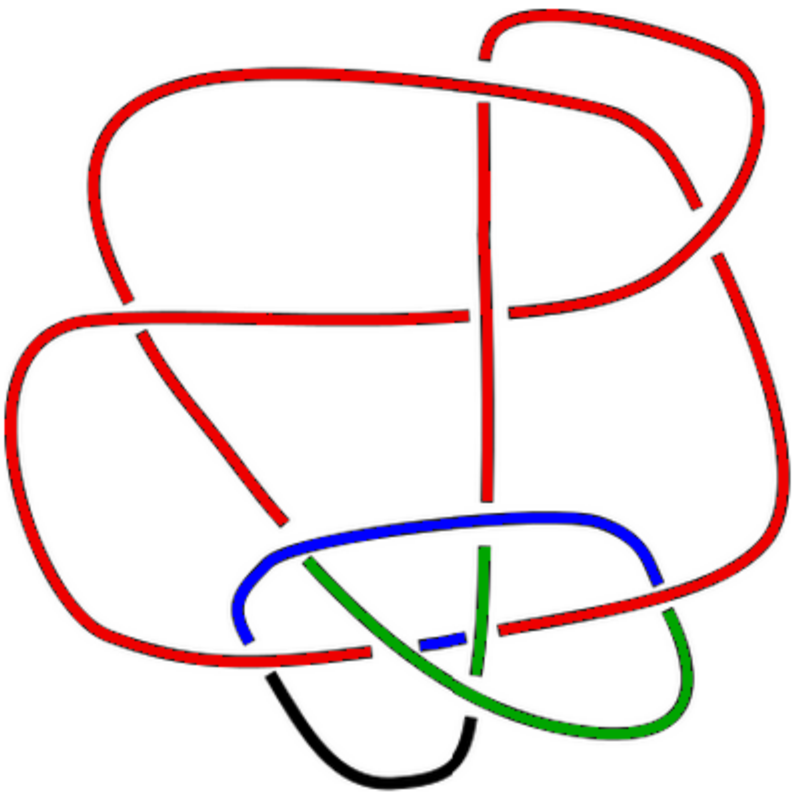


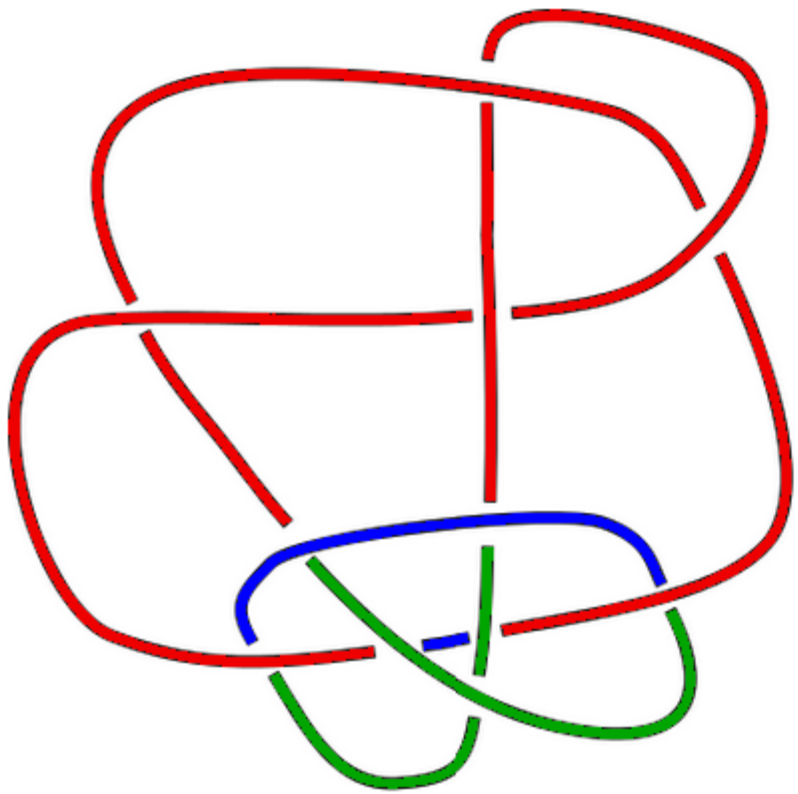




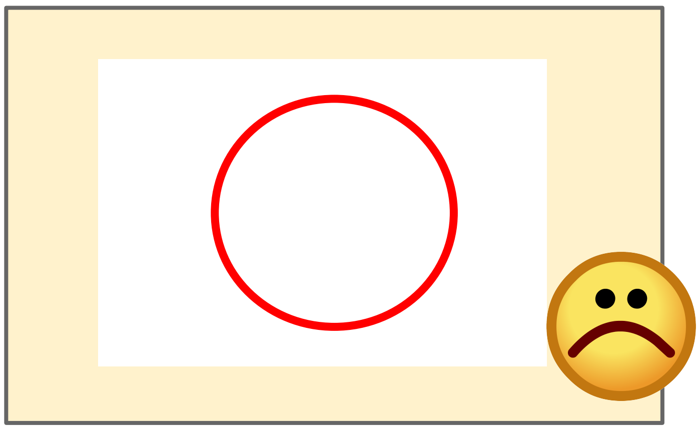


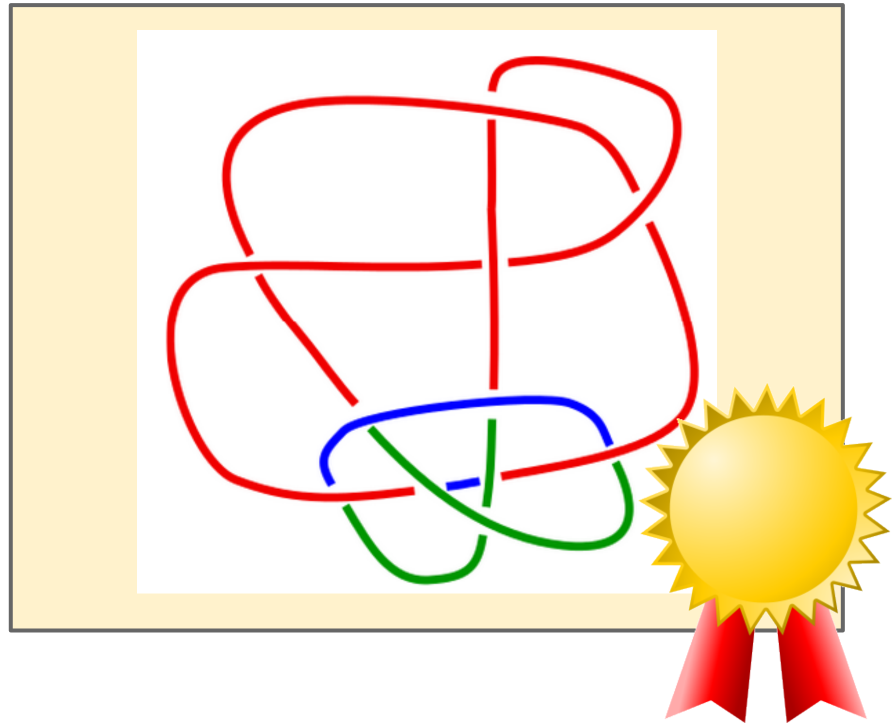


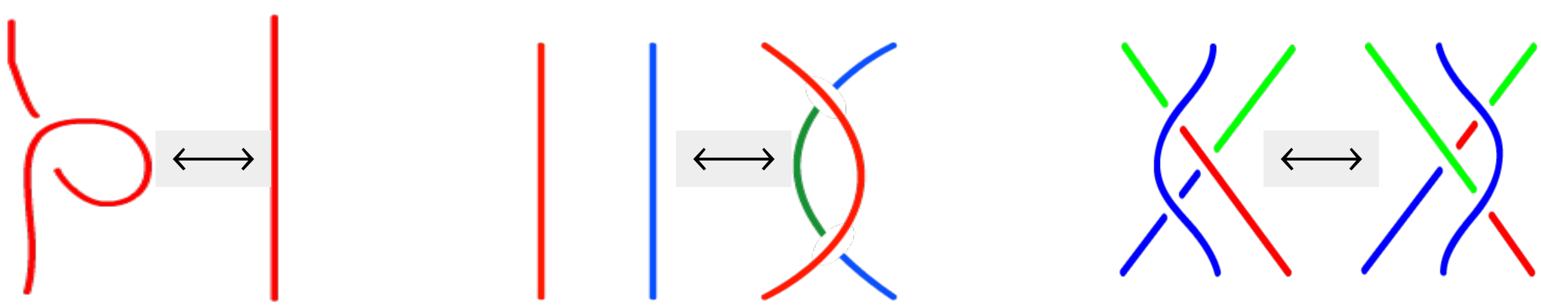




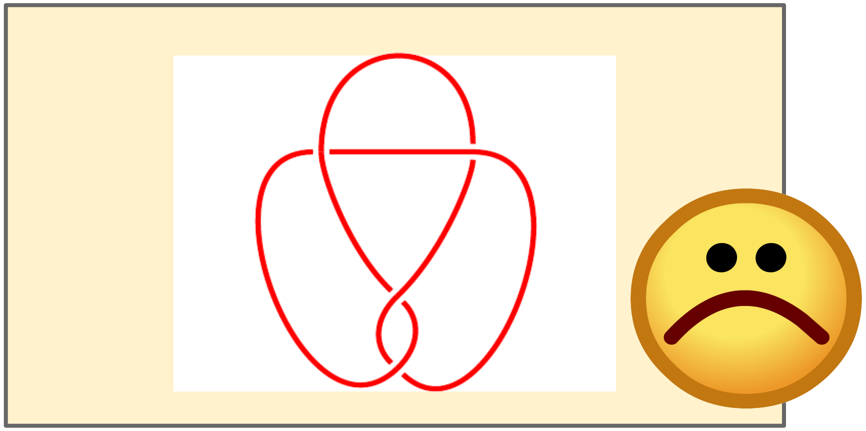
A tri-coloring is a "certificate of knottedness."

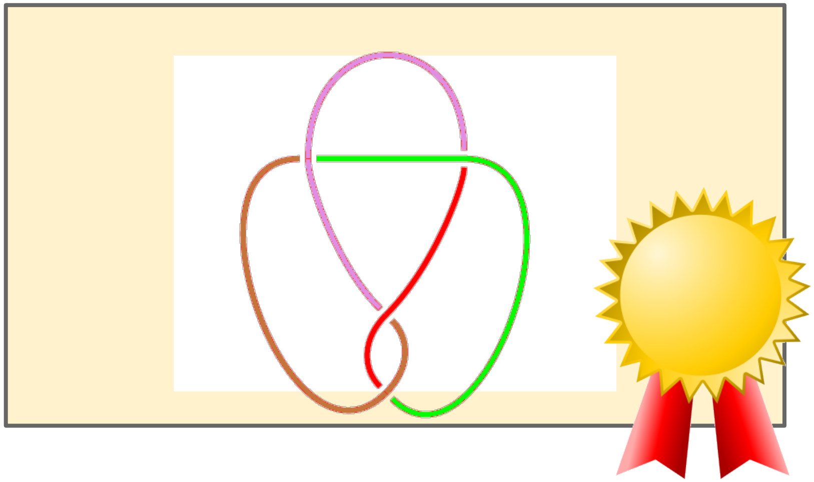




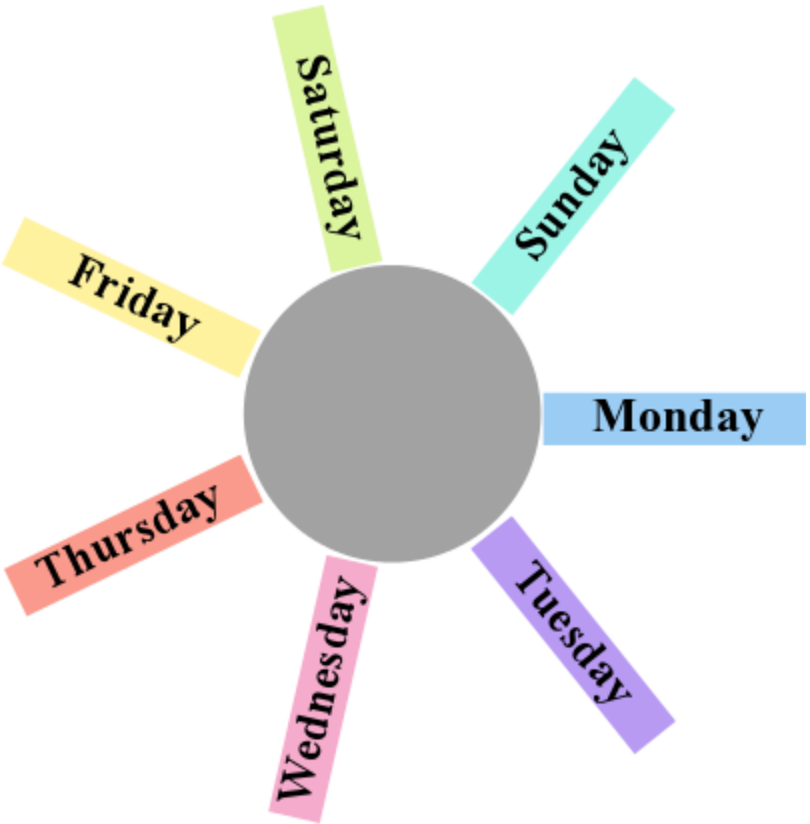


But some bona fide knots aren't tri-colorable





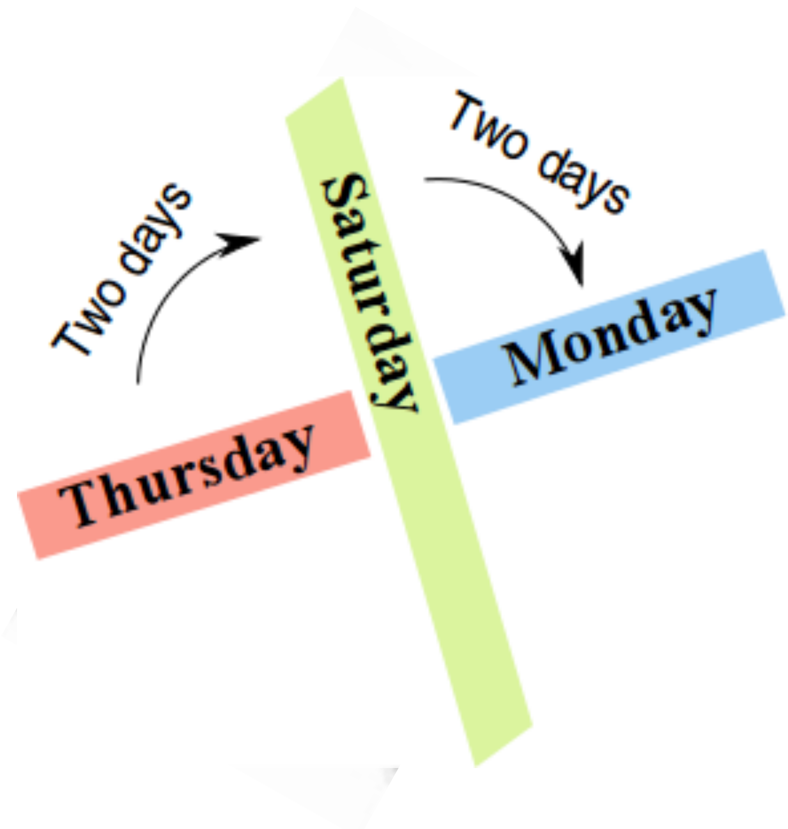
What about more colors?

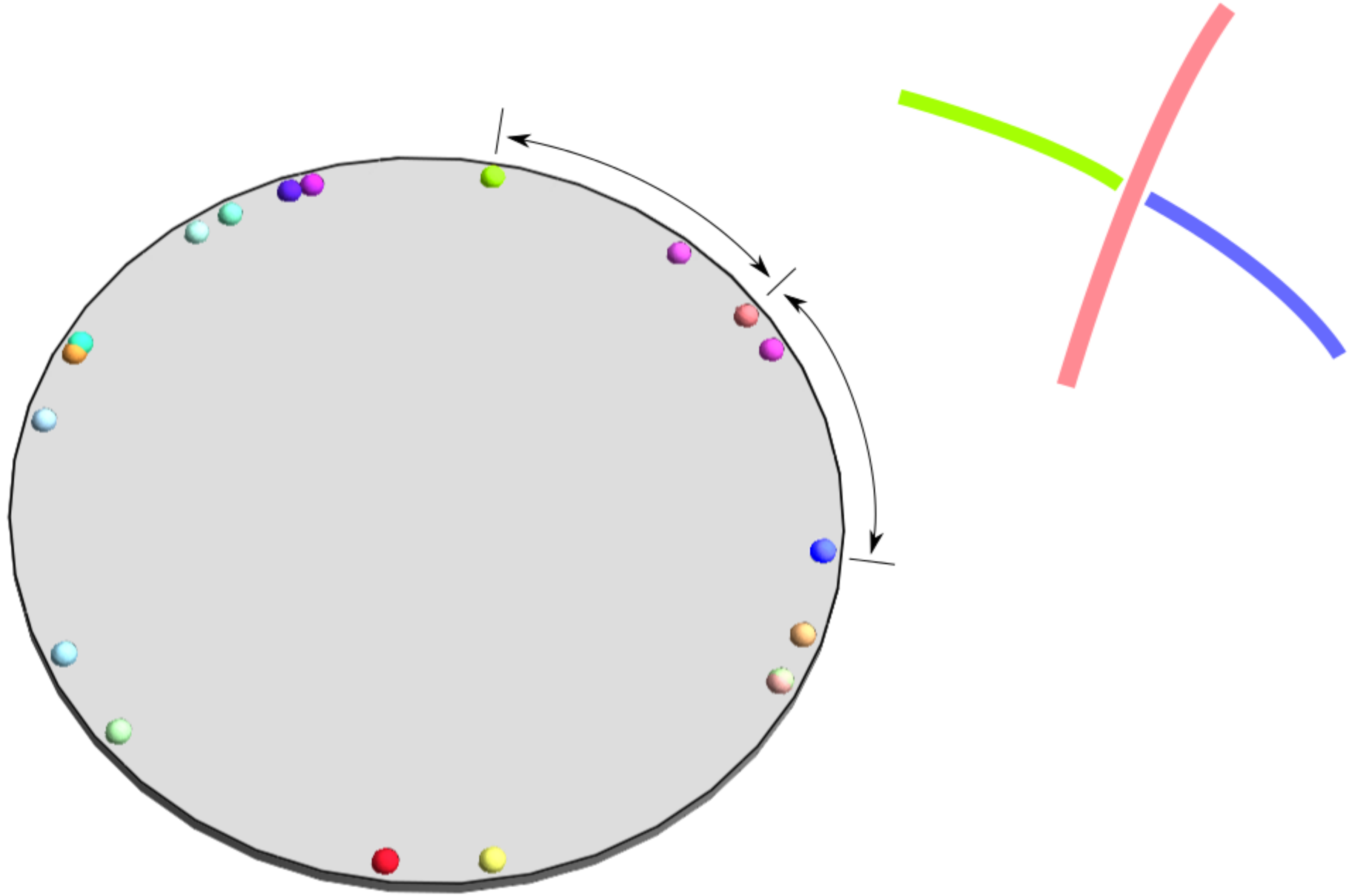


We can use seven colors, arranged in a circle ...

Or twelve colors, arranged in a circle …

Coloring rules

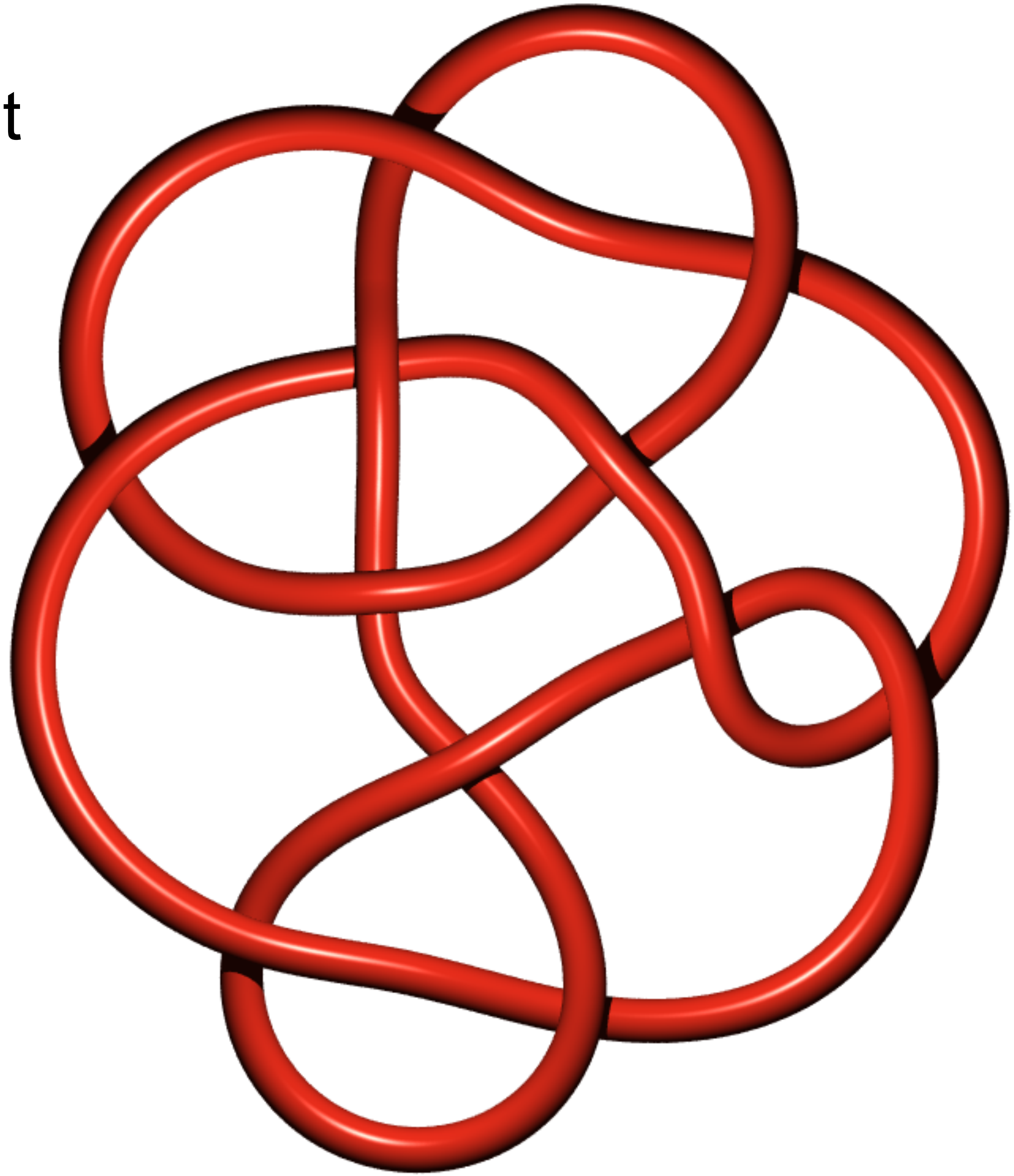




The Conway knot

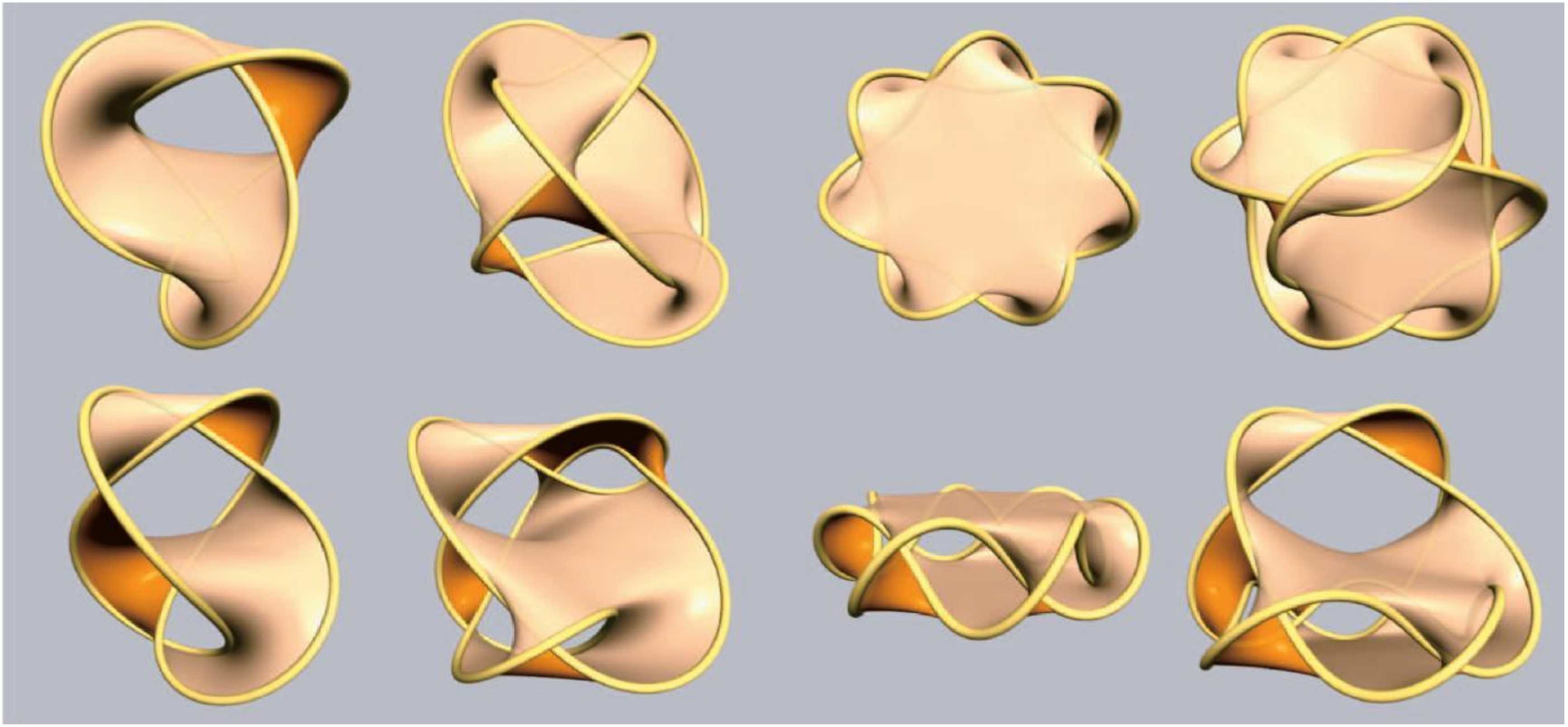
11n34

It only has the trivial coloring



Other Invariants.

Every knots is the boundary of some oriented surface in 3-space.



Seiferts Algorithm.

1. Orient the knot or link.
2. Resolve each crossing in an orientation preserving manner.
3. The result is a collection of oriented nested circles in the plane. For each bounded region lift the corresponding disk up out of the plane with height depending on how nested it is.
4. Now join the disks back up with half-twisted bands.

Knot Genus

Recall that oriented surfaces are classified by the either Euler characteristic and the number of boundary components (assume the surface is connected). For closed surfaces this is an even integer written **2-2g** where **g** is the genus of the surface The Euler characteristic of Seifert surface of a knot is thus odd **1-2g.**

The genus of a knot is minimum g occuring for all surfaces (oriented embedded connected) bounding K.

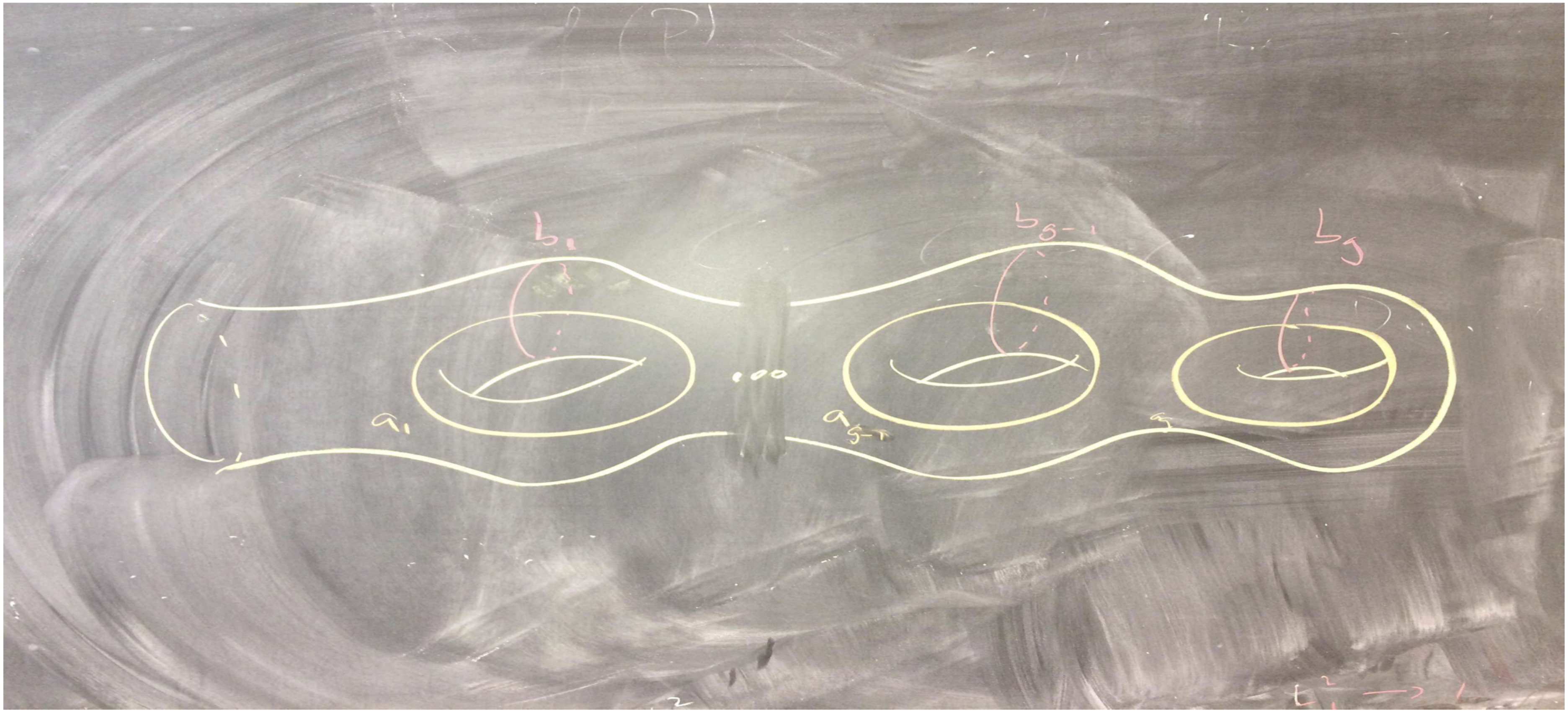
Certificate of Unknottedness

Note that K is an unknot if and only if K bounds a (smoothly) embedded disk i.e. has genus 0.

One direction is easy. The standard unknot bounds the standard disk. In the other direction if K bounds a disk. Shrink K along concentric circles in the disk until it is very small. Because the disk is smooth the new curve is basically a small ellipse.

The Seifert Matrix

Take the following collection of curves on a Seifert Surface.



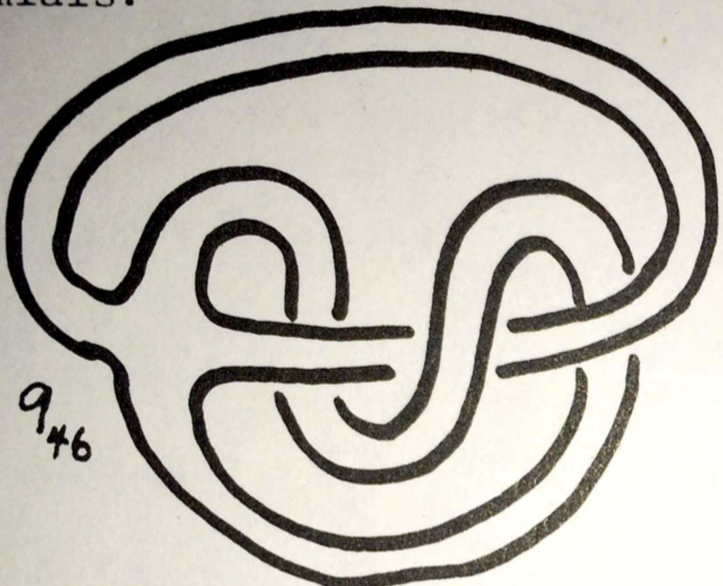
Push each of them off according to both the positive and negative normal to the surface.

We get a collection of curves in the complement of the Siefert surface $a^+_1, a^-_1, b^+_1, b^-_1$, etc. Form the matrix V whose entries are linking numbers of the minus curves with the plus curves.

Compute:

$$\det(V - tV^T)$$

lthough the distinction is lost when we pass to Alexander ideals and
olynomials.



9_{46}

$$V = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$V^T - tV = \begin{bmatrix} 0 & 1-2t \\ 2-t & 0 \end{bmatrix}$$

$$\Delta(t) = 2 - 5t + 2t^2$$

6_1

$$V = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

$$V^T - tV = \begin{bmatrix} 0 & 1-2t \\ 2-t & t-1 \end{bmatrix}$$

$$\Delta(t) = 2 - 5t + 2t^2$$

It turns out that this determinant, called the Alexander polynomial is a knot invariant up to multiplication by $+-t^n$. That is, it does not depend on the particular Siefert surface nor the way the curves used are drawn on that surface. Note that the spread in the degree of the Alexander polynomial gives a bound on the genus.

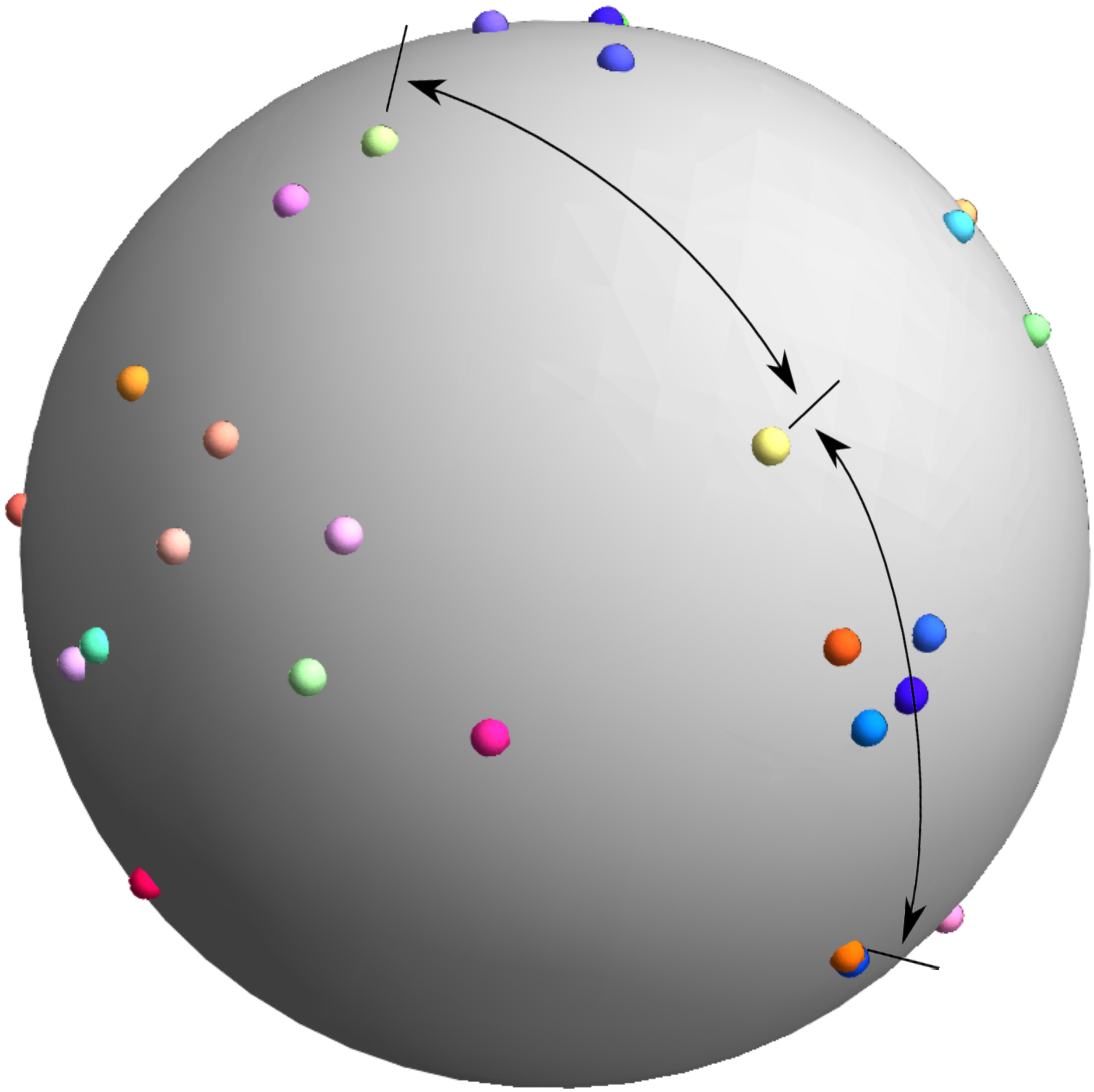
$$\tfrac{1}{2}\ \text{degree-spread}(\text{Alex}(K)) > \text{genus}(K)-1$$

Another computational tool for the Alexander polynomial. Skein relations.

The Alexander Polynomial of a the Conway knots is 1. It can’t distinguish it from and unknots.

It turns out that a variant of coloring
can detect non-trivial knotting reliably.

Two sphere colorings!

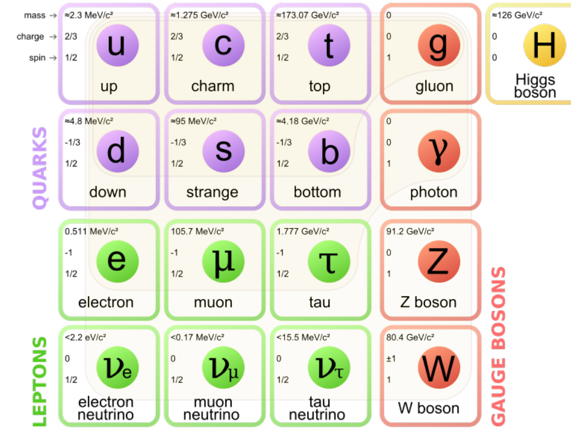


Sphere-colorings of knots

Theorem (Kronheimer-Mrowka, 2010):
Every knot except the unknot can be given a non-trivial sphere-coloring.

We can use colors arranged on the sphere to provide a certificate of knotedness for every bona-fide knot.

Techniques we use



- gauge theory
- Yang-Mills equations

Origins in theoretical physics, the description of fundamental particles (quarks, leptons) and their interactions

Karen Uhlenbeck

Cliff Taubes

Simon Donaldson

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