SPHERE PACKING PROBLEM SET

IAP MATH LECTURE SERIES

Recall that

$$D_n = \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_1 + \cdots + x_n \text{ is even}\},$$

while E_8 is the union of D_8 and the translation of D_8 by the vector $(1/2, 1/2, \dots, 1/2)$. Equivalently, E_8 consists of all points $(x_1, \dots, x_8)$ with two properties:

(1) either all coordinates are integers or all coordinates are integers plus $1/2$, and
(2) the coordinate sum $x_1 + \cdots + x_8$ is an even integer.

Problems 1–3 should be more doable, although they may take some calculation or thought. Problems 4 and 5 have simple solutions but take more of an insight.

1. How many spheres are tangent to each sphere in the D_n packing?

2. What about E_8?

4. Compute the volume of a four-dimensional ball of radius r. (You can use multivariate calculus without worrying about rigorous justification of volume integrals in $\mathbb{R}^4$.) What is the density of the D_4 sphere packing in $\mathbb{R}^4$?

5. Recall that the Platonic solids are the regular tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Which ones can occur as cross sections of a four-dimensional hypercube? How do they occur, and why can't the others?

6. Suppose $P_1, \dots, P_{10}$ are any ten points in the plane. Show that there exist closed unit disks $D_1, \dots, D_{10}$ that cover them, such that these disks have disjoint interiors. (Covering means each point P_i in contained in some disk D_j, and disjoint interiors means that the disks can be tangent, but they can't overlap any more than that.)