Problem Set: Euler's Elastica.

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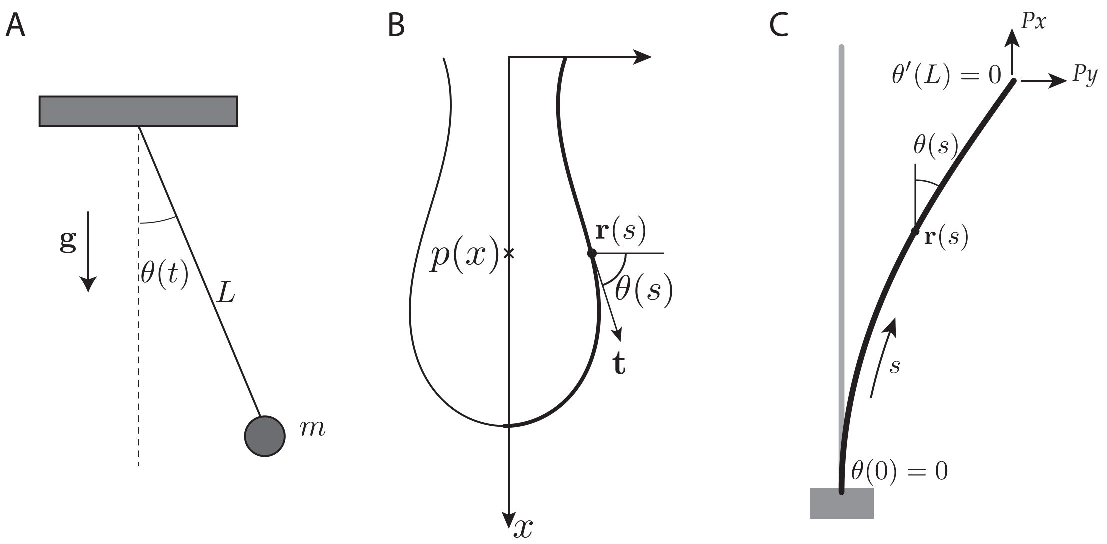

1. Pendulum

What is the energy E of a pendulum expressed in term of the angle θ and its time derivative (see Fig-A)? Using the software of your choice graph and classify the orbits of the pendulum in the phase plane $(\theta, \dot{\theta})$ when varying E. Compare those solutions with the harmonic oscillator. TIP: One may use the ContourPlot tool in Mathematica$^{\text{TM}}$.

2. The shape of a 2D-pendant drop

Derive the equation describing the shape of pendant drop surrounded by air using the set of coordinates defined in Fig-B. We recall that:

- when crossing a curved interface the pressure is subject to a discontinuity Δp which is proportional to surface tension γ and curvature κ. $\Delta p = \gamma\kappa$,
- in a liquid at rest the pressure gradient is $\nabla p = \rho \mathbf{g}$ where $\mathbf{g}$ is the acceleration of gravity and ρ the fluid density.

TIP: Express $p(x)$ using two different methods and derive the result.

3. Linear response of the Euler elastica

Let us consider Euler's Elastica (see Fig-C.) and assume that P_x and P_y are small enough so that $\theta \ll 1$. Derive the linearized equation describing $\theta(s)$. Reconstruct and plot the Elastica shape. Plot the value of the moment $m(s)$. We recall the equations for the equilibrium of an elastic rod neglecting the action of gravity:

$$\frac{d\mathbf{n}}{ds} = \mathbf{0} \tag{1}$$

$$\frac{dm}{ds} + \mathbf{t} \times \mathbf{n} = \mathbf{0} \tag{2}$$

$$\mathbf{m}(s) \cdot \mathbf{e}_z = m(s) = B\theta' \tag{3}$$

where $\mathbf{n}(s)$, $\mathbf{m}(s)$, θ' and B are the rod internal stress, internal moment, curvature, and bending stiffness, respectively.