Problems for Lecture #1 on p-adic numbers

1. Prove that $\mathbb{Q}_p$ is not an ordered field; that is, prove that there is no partition of $\mathbb{Q}_p$ into sets P, $\{0\}$, and $-P = \{-x : x \in P\}$ such that P is closed under addition and multiplication.

2. Let x, y, z be three distinct elements of a field with a nonarchimedean absolute value $\| \ \|$.
   Prove that at most two of the real numbers $\|x-y\|, \|y-z\|, \|z-x\|$ are distinct.
   (in other words, every nonarchimedean "triangle" is isosceles).

3. Let k be field with a nonarchimedean absolute value $\| \ \|$. A *ball* in k is any set of the form
   $$B(x,r) := \{y : \|x-y\| < r\}.$$
   Prove that for every $x \in k$ and $y \in B(x,r)$ we have $B(y,r) = B(x,r)$.
   (in other words, every point in a ball is "at the center").
   Conclude that if two balls intersect then one contains the other.

4. Let k be field with a nonarchimedean absolute value $\| \ \|$.
   Prove that a sequence (x_n) of elements of k is Cauchy if and only if $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
   (this is not true in $\mathbb{R}$, but it is in $\mathbb{Q}_p$, which makes p-adic analysis a lot easier).

5. Prove that the p-adic representation of $x \in \mathbb{Q}_p$ is eventually periodic if and only if $x \in \mathbb{Q}$.

6. Let $| \ |_\infty$ denote the archimedean absolute value on $\mathbb{Q}$, and for each prime p let $| \ |_p$ denote the p-adic absolute value. Prove that the *product formula*
   $$\prod_{p \le \infty} |x|_p = 1$$
   holds for every nonzero $x \in \mathbb{Q}$, where the index p ranges over all primes and the symbol ∞.