Math 18.02, r11
Problems #20
July 1, 2013

1. *Convert the following points from Cartesian to spherical coordinates:* $(1,1,0)$, $(0,1,1)$, $(1,1,1)$. *What about* $(0,0,-3)$*?*

   $(\sqrt{2}, \pi/4, \pi/2)$, $(\sqrt{2}, \pi/2, \pi/4)$, $(\sqrt{3}, \pi/4, \cos^{-1}(1/\sqrt{3}))$, $(3, 0, \pi/2)$. Note that θ isn't really well-defined for the last one of these. It's just like in polar coordinates at the origin, where any value of θ gives the same point.

   Convert the following points from spherical to Cartesian coordinates: $(1,\pi/2,\pi/2)$, $(2,\pi/6,\pi/3)$, $(3,-\pi/2,3\pi/4)$.

   $(0,1,0)$, $(3,\sqrt{3}/2,1)$, $(0,-3/\sqrt{2},-3/\sqrt{2})$.

2. *What regions are described by these constraints in spherical coordinates?*

   (a) $\rho = 2$
   (b) $\theta = \pi/2$
   (c) $0 \leq \phi \leq 3\pi/4$
   (d) $0 \leq \theta \leq \pi/4$
   (e) $1 \leq \rho \leq 2$, $0 \leq \phi \leq \pi/2$

   (a) Sphere of radius 2.
   (b) Plane perpendicular to the xy-plane, between the xz-plane and the yz-plane.
   (c) Everything except a downward-pointing cone of vertex angle $\pi/2$.
   (d) An eighth of a cylinder parallel to the z-axis.
   (e) The upper half of a spherical shell with inner radius 1 and outer radius 2.

3. *Set up bounds of integration for the following regions in spherical coordinates.*

   (a) *The part of the unit sphere in the first octant.*

   $$\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{\rho=0}^{1} d\rho\, d\theta\, d\phi$$

   We want ϕ to go from 0 to $\pi/2$ to ensure that the z coordinate is positive. θ in 0 to $\pi/2$ is the region whose xy part is in the xy-plane.

   (b) *The region between the planes* $z = 1$ *and* $z = 2$.

   One way to do this is to remember that $z = \rho\cos\phi$. So $1 \leq z \leq 2$ means $1/\cos\phi \leq \rho \leq 2/\cos\phi$.

   $$\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=1/\cos\theta}^{2/\cos\theta} d\rho\, d\theta\, d\phi$$

(c) *The region between the planes $x = 1$ and $x = 2$.*

Use the same trick to change coordinates. $x = \rho \cos\theta \sin\phi$, so we want $\sec\theta \csc\phi \leq \rho \leq 2 \sec\theta \csc\phi$.

$$\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=\sec\theta\csc\phi}^{2\sec\theta\csc\phi} d\rho\, d\theta\, d\phi$$

(d) *(5B1c) The region bounded by a sphere of radius $\sqrt{2}$ passing through the origin, and the cone $x^2 + y^2 = z^2$.*

To land inside this cone we need $\phi \leq \pi/4$. The other bounds are easy to see. In this case the order isn't going to matter.

$$\int_{\phi=0}^{\pi/4} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{\sqrt{2}} d\rho\, d\theta\, d\phi$$

4. *Compute the volume of a sphere of radius a.*

$$\begin{aligned} V &= \int_{\rho=0}^{a} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho \\ &= 2\pi \int_{\rho=0}^{1} \int_{\phi=0}^{\pi} \rho^2 \sin\phi\, d\phi\, d\rho \\ &= 4\pi \int_{\rho=0}^{a} \rho^2\, d\rho \\ &= \frac{4\pi a^3}{3} \end{aligned}$$

5. *Find the center of mass of a solid hemisphere of radius a.*

$$\begin{aligned} \bar{z} &= \frac{1}{M} \int_{\rho=0}^{a} \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} (\rho\cos\phi)\rho^2 \sin\phi\, d\theta\, d\phi\, d\rho \\ &= \frac{2\pi}{M} \int_{\rho=0}^{a} \int_{\phi=0}^{\pi/2} \rho^3 \cos\phi \sin\phi\, d\phi\, d\rho \\ &= \frac{\pi}{M} \int_{\rho=0}^{a} \rho^3\, d\rho = \frac{\pi a^4}{4M} \end{aligned}$$

But $M = 2\pi a^3/3$, so

$$\bar{z} = \frac{3}{2\pi a^3} \frac{\pi a^4}{4} = \frac{3a}{8}.$$

(see next recitation for the flux problems)

6. *Write down equations for these vector fields in three dimensions:*

(a) *A field pointing directly away from the origin which has magnitude* 1 *at every point.*

(b) *The gravitational field of a point mass at* $(1, 0, 0)$.

(c) *A field which circles clockwise around the* x*-axis, with length equal to the distance from the axis.*