Math 18.02, r11
Problems #19
July 1, 2013

1. *Give bounds of integration for the following three-dimensional regions, using Cartesian coordinates. Compute the volumes of these regions by integrating the function* 1.

   (a) *A rectangular prism in the first octant with sides of length* 3, 4, *and* 5, *parallel to the* x-, y-, *and* z-*axes.*

   $$\int_{z=0}^{5}\int_{y=0}^{4}\int_{x=0}^{3} 1\,dx\,dy\,dz = 60.$$

   (b) *The region over a triangle with vertices at* $(0,0)$, $(2,0)$, *and* $(0,4)$ *beneath the graph of* $z = x^2 + y^3$.
   The hypotenuse of the triangle has equation $y = 4-2x$, so the range on y will be 0 to $4-2x$ (this is the same way we'd set up a double integral over the triangle). Then z should range from 0 to x^2+y^3.

   $$\begin{aligned} V &= \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{x^2+y^3} 1\,dz\,dy\,dx \\ &= \int_{x=0}^{2}\int_{y=0}^{4-2x} x^2+y^3\,dy\,dz = \int_{x=0}^{2} (4-2x)x^2 + \frac{(4-2x)^3}{3}\,dx = \frac{424}{15}. \end{aligned}$$

   (c) *The region bounded by the three coordinate planes and the plane* $x+y+z=1$.
   This plane intersects the axes at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. So its base in the xy-plane is a triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. Our x and y bounds will just give a parametrization of this triangle in the xy-plane. We want z to range from 0 up to where it hits the plane, which is at $1-x-y$:

   $$\begin{aligned} V &= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 1\,dz\,dy\,dx \\ &= \int_{x=0}^{1}\int_{y=0}^{1-x} 1-x-y\,dy\,dx \\ &= \int_{x=0}^{1} (1-x)^2 - \frac{(1-x)^2}{2}\,dx = \frac{1}{6}. \end{aligned}$$

   (d) *A sphere of radius* a, *centered at the origin.*
   The base is a circle of radius a, which we know how to parametrize, just like a circle in 2D, we want z to range from $-\sqrt{a^2-x^2-y^2}$ to $\sqrt{a^2-x^2-y^2}$ (since the equation for the sphere is $x^2+y^2+z^2=a^2$).

   $$\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} 1\,dz\,dy\,dx$$

   This isn't a very friendly integral, so let's postpone compute the volume of the sphere until later when we cover spherical coordinates.

(e) *The region between the xy-plane and the surface $z = 1 - x^2 - y^2$.* The base is the area where $z > 0$, which is $x^2 + y^2 < 1$, the unit disk. So

$$V = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{1-x^2-y^2} 1\, dz\, dy\, dz.$$

2. *Express the points $(1,1,3)$ and $(0,2,4)$ in cylindrical coordinates.* We just need to treat the first two as 2D Cartesian coordinates and convert to polar, while leaving the last one the same. This gives $(\sqrt{2}, \pi/4, 3)$ and $(2, \pi/2, 4)$.

3. *Give bounds of integration for the following three-dimensional regions, using cylindrical coordinates:*

   (a) *A cylinder of radius 2 and height 4, centered at the origin.*

   This is the basic example for cylindrical coordinates. We use

$$\int_{r=0}^{2} \int_{\theta=0}^{2\pi} \int_{h=-2}^{2} 1\, r\, dh\, d\theta\, dr.$$

   (b) *A sphere of radius a, centered at the origin.*

   Here the height depends on the radius. The formula $x^2 + y^2 + z^2 = a^2$ turns into $r^2 + h^2 = a^2$, so h is from $-\sqrt{a^2 - r^2}$ to $\sqrt{a^2 - r^2}$.

$$\int_{r=0}^{a} \int_{\theta=0}^{2\pi} \int_{h=-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} 1\, r\, dh\, d\theta\, dr.$$

   (c) *A cone with base a circle of radius 1 and height 3.*

   The height decreases linearly as the radius increases from 0 to 1. Since it starts with $h = 3$ at $r = 0$, and ends with $h = 0$ and $r = 1$, we must have $h = 3 - 3r$. Thus

$$\int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{h=0}^{3-3r} 1\, r\, dh\, d\theta\, dr.$$

   (d) *The region between the xy-plane and the surface $z = 1 - x^2 - y^2$.*

   Same as before, except we rewrite everything in cylindrical

$$\int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{h=0}^{\sqrt{1-r^2}} 1\, r\, dh\, d\theta\, dr.$$

4. (a) *(5A-3) Find the center of mass of the tetrahedron from 1(c).*

   We use the same bounds as in the first part, but now we need to integrate the function x to get the x-coordinate of the center of mass (similarly for the other variables, but the answers will all be the same). Compute

$$\begin{aligned}\bar{x} &= \frac{1}{M}\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} x\,dz\,dy\,dx \\ &= 6\int_{x=0}^{1}\int_{y=0}^{1-x} x(1-x-y)\,dy\,dx \\ &= 6\int_{x=0}^{1} x((1-x)^2 - \frac{(1-x)^2}{2})\,dx = \frac{1}{4}.\end{aligned}$$

Therefore the center of mass is $(1/4, 1/4, 1/4)$.

(b) *Compute the moment of inertia of the sphere of radius a about the z-axis (use cylindrical coordinates). Assume the density is uniform 1.*

The bounds are the same as before. We want to integrate the function which gives the squared distance from (r, θ, h) to the z-axis, which is of course just r^2.

$$I = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{h=-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^2\, r\,dh\,d\theta\,dr.$$

If you work it out (a bit of trouble, but doable), you'll get $I = 8\pi a^5/15$.

You might remember that $I = 2/5\,MR^2$ from a physics course. Since $M = 4\pi/3\,a^a$ and $R = a$, this gives the same answer that we just computed directly.