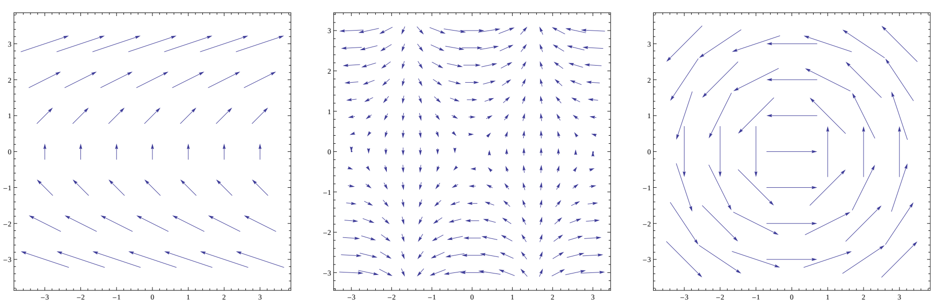
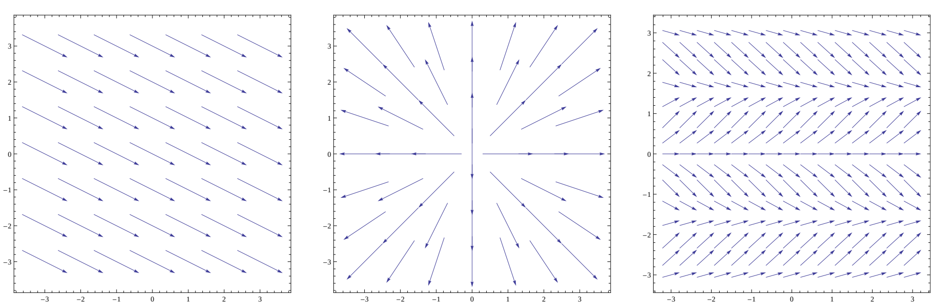
Math 18.02, r11
Problems #14
June 30, 2013

1. *Sketch the following vector fields:*
   (a) $\vec{F}_1(x,y) = y\hat{\imath} + \hat{\jmath}$
   (b) $\vec{F}_2(x,y) = y\cos x\,\hat{\imath} + \sin x\,\hat{\jmath}$
   (c) $\vec{F}_3(x,y) = -\frac{y}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{x}{\sqrt{x^2+y^2}}\hat{\jmath}$

   Here they are, courtesy of Mathematica:



2. *Give the equation defining each of the vector fields illustrated below:*



   The first one is a constant vector field, since all of the arrows are the same. Looking at the slope, we might guess $\vec{F}(x,y) = 2\hat{\imath} + \hat{\jmath}$. (It's hard to tell for sure from the picture)

   This is a vector field which at (x,y) points in the direction from the origin to (x,y). Thus it's some multiple of $\vec{F}(x,y) = x\hat{\imath} + y\hat{\jmath}$. Since all the lengths of the vectors in the picture are 1, it must be normalized so $\vec{F}(x,y) = \frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath}$.

   The third is a bit tricky. The first thing to notice is that the horizontal part is constant, so the field is of the form $\vec{F}(x,y) = \hat{\imath} + N(x,y)\hat{\jmath}$. Notice that $N(x,y)$ appears not to depend on x at all: for any value of x, as we change y the field changes in exactly the same way. Moreover, it seems to be periodic in y (well, this might be hard to see since I didn't include a big enough region – trust me here). The vertical part vanishes at 0, $\pi/2$, $\pi, \ldots$. So it looks like a pretty good guess is that the field is $\vec{F}(x,y) = \hat{\imath} + \sin(2y)\hat{\jmath}$.

3. *(4B-2) Evaluate the integral of $\vec{F} = x\hat{\imath} + y\hat{\jmath}$ along the path C which goes once counter-clockwise around a circle of radius a centered at the origin. First argue geometrically, and then check your answer by directly computing the integral.*

   Well, $\frac{d\vec{r}}{dt}$ is always tangent to the circle (this is the velocity). But $F(\vec{r}(t))$ is a radius. The two are perpendicular, so their dot product is 0, and the line integral must be 0. It's not much harder to make this precise. We parametrize the path via $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$, and so the velocity is $d\vec{r}/dt = \langle -a\sin t, a\cos t\rangle$. The vector field is $\vec{F}(\vec{r}(t)) = \langle a\cos t, a\sin t\rangle$. So

$$\int_C F d\vec{r} = \int_{t=0}^{2\pi} \langle a\cos t, a\sin t\rangle \cdot \langle -a\sin t, a\cos t\rangle \; dt = \int_{t=0}^{2\pi} 0\, dt = 0.$$

4. *For the first and second vector fields of problem 1, compute the integral along three different paths from $(0,0)$ to $(1,1)$:*

   (a) *C_1, a straight line from $(0,0)$ to $(1,1)$*

   (b) *C_2, a line from $(0,0)$ to $(1,0)$ and then to $(1,1)$.*

   (c) *C_3, along the parabola $y = x^2$*

   The paths are parametrized by $\vec{r}(t) = \langle t, t\rangle$, $\vec{r}(t) = \langle t, t^2\rangle$, and $vecr(t) = \langle t, 0\rangle$ followed by $\vec{r}(t) = \langle 1, t\rangle$. The velocity vectors are respectively $\langle 1, 1\rangle$, $\langle 1, 2t\rangle$, and $\langle 1, 0\rangle$ followed by $\langle 0, 1\rangle$. For the vector field $\vec{F}(x,y) = y\hat{\imath} + \hat{\jmath}$, we compute the integrals as

$$\int_{C_1} \vec{F}\cdot\vec{dr} = \int_{t=0}^{1} \langle t, 1\rangle \cdot \langle 1, 1\rangle \; dt = \int_{t=0}^{1} (t+1)\, dt = \frac{3}{2},$$
$$\int_{C_2} \vec{F}\cdot\vec{dr} = \int_{t=0}^{1} \left\langle t^2, 1\right\rangle \cdot \langle 1, 2t\rangle \; dt = \int_{t=0}^{1} (t^2+2t)\, dt = \frac{4}{3},$$
$$\int_{C_3} \vec{F}\cdot\vec{dr} = \int_{t=0}^{1} \langle 0, 1\rangle \cdot \langle 1, 0\rangle \; dt + \int_{t=0}^{1} \langle t, 1\rangle \cdot \langle 0, 1\rangle \; dt = 0 + 1 = 1.$$

   For the second field, $\vec{F}(x,y) = y\cos x\,\hat{\imath} + \sin x\,\hat{\jmath}$.

$$\begin{aligned}\int_{C_1} \vec{F}\cdot\vec{dr} &= \int_{t=0}^{1} \langle t\cos t, \sin t\rangle \cdot \langle 1, 1\rangle \; dt = \int_{t=0}^{1} (t\cos t + \sin t)\, dt \\ &= (t\sin t)\big|_0^1 = \sin 1, \\ \int_{C_2} \vec{F}\cdot\vec{dr} &= \int_{t=0}^{1} \left\langle t^2\cos t, \sin t\right\rangle \cdot \langle 1, 2t\rangle \; dt = \int_{t=0}^{1} (t^2\cos t + 2t\sin t)\, dt \\ &= \left(t^2\sin t\right)\big|_0^1 = \sin 1, \\ \int_{C_3} \vec{F}\cdot\vec{dr} &= \int_{t=0}^{1} \langle 0, \sin t\rangle \cdot \langle 1, 0\rangle \; dt + \int_{t=0}^{1} \langle t\cos 1, \sin 1\rangle \cdot \langle 0, 1\rangle \; dt \\ &= 0 + \sin \int_0^1 \sin 1\, dt = \sin 1.5\end{aligned}$$

5. *One of the vector fields from the previous question is conservative. Which one? Find a function $f(x, y)$ of which it is the gradient, and evaluate the above integral using the fundamental theorem for line integrals.*

   We got the same integral over all three curves when integrating the second field, suggesting that it is probably a gradient field. In fact, it is the gradient field of $f(x, y) = y \sin x$, and the fundamental theorem of calculus for line integrals implies that

$$\int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(0,0) = \sin 1 - 0 = \sin 1,$$

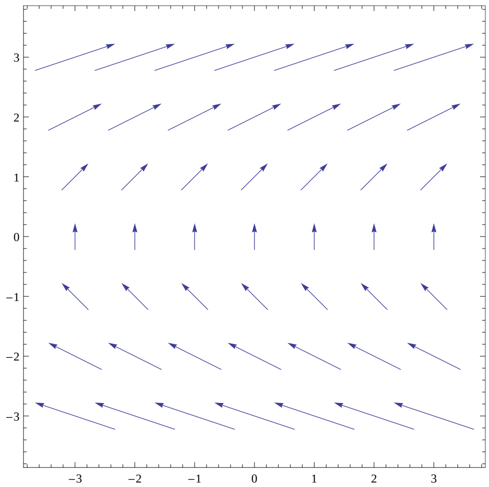
   for any path C between $(0, 0)$ and $(1, 1)$, including the three used above.

6. *What is the gradient field associated with the function $f(r, \theta) = r \log r$?*

   One way to do this is with the chain rule:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} = (1 + \log r)(x/r) + 0\frac{\partial \theta}{\partial x} = (1 + \log r)(x/r) \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} = (1 + \log r)(y/r) + 0\frac{\partial \theta}{\partial x} = (1 + \log r)(y/r).\end{aligned}$$

   You could now plug in the usual formulas for polar coordinates to express the answer in terms of only x and y, but leaving it this way is OK too.



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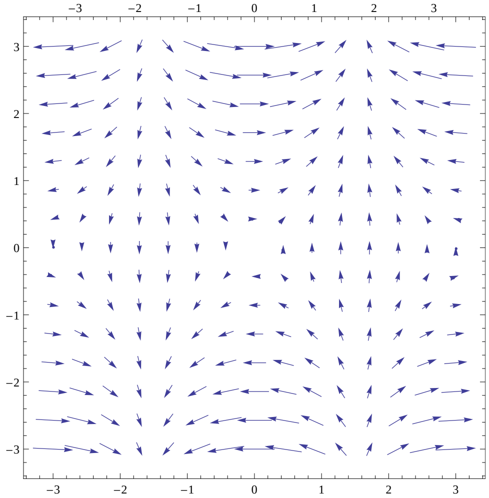

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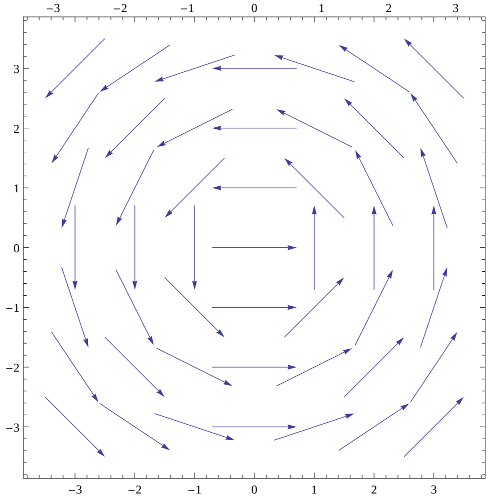

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