Math 18.02, r11
Problems #13
July 1, 2013

1. *Evaluate the integral*
$$\int_{x=1}^{2}\int_{y=0}^{x}\frac{1}{(x^2+y^2)^{3/2}}\,dy\,dx$$
by converting to polar coordinates.

2. *Integrate* $f(x,y)=x^2-y^2$ *over the square with vertices* $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$. *Use the coordinate change* $u=x+y$, $v=x-y$.

3. *Find the area of a parallelogram with sides* $y-x=0$, $y-x=2$, $3x-y=0$, $3x-y=4$.

   To find the area of anything, we need to integrate the function $f(x,y)=1$ over it. In order to make this easier, we'll use the change of coordinates $u=y-x$, $v=3x-y$. In these coordinates, the sides of the parallelogram are just $v=0$, $v=4$, $u=0$, and $u=2$. Solving the linear equations, this gives us $x=(u+v)/2$ and $y=(3u+v)/2$, so the Jacobian is $\left|\begin{vmatrix} 1/2 & 1/2 \\ 3/2 & 1/2 \end{vmatrix}\right| = |1/4-3/4| = 1/2$.

   Now we integrate to find the area:
$$A=\int_{v=0}^{4}\int_{u=0}^{2}1\left(\frac{1}{2}\,du\,dv\right)=4.$$

4. *Find the area of the region defined by* $x^{2/3}+y^{2/3}=1$ *in the first quadrant.*

   Again we integrate the function 1. Take $u=x^{2/3}$, $v=y^{2/3}$. Our region is bounded by the three curves $x=0$, $y=0$, $x^{2/3}+y^{2/3}=1$. In terms of u and v, these are respectively $u=0$, $v=0$, $u+v=1$, which is just a triangle. We have $x=u^{3/2}$ and $y=v^{3/2}$, and so the Jacobian is $J=\begin{vmatrix} 3\sqrt{u}/2 & 0 \\ 0 & 3\sqrt{v}/2 \end{vmatrix}$. So the integral is
$$\int_{u=0}^{1}\int_{v=0}^{1-u}\frac{9}{4}\sqrt{uv}\,dv\,du=\frac{9}{4}\frac{\pi}{24}=\frac{3\pi}{32}.$$

   It's a bit of work to actually do the integral, but I omit the calculation.

5. *(14.9.12) Integrate* $f(x,y)=\frac{1}{(x^2+y^2)^2}$, *over the region in the first quadrant bounded by the circles* $x^2+y^2=2x$, $x^2+y^2=6x$, *and* $x^2+y^2=2y$, $x^2+y^2=8y$. *Use* $u=2x/(x^2+y^2)$ *and* $v=2y/(x^2+y^2)$.

   First let's work out the limits of integration in the new coordinates. The circle $x^2+y^2=2y$ translates to $\frac{2y}{x^2+y^2}=1$, which is just the condition $v=1$. In the same way, the other three boundaries are expression in terms of u and v as $u=1$, $u=1/3$, $v=1/4$. So we're just going to be integrating over a rectangle. The hard part is to compute the Jacobian and expression the function in our new coordinates. Observe that
$$u^2+v^2=\frac{4x^2}{(x^2+y^2)^2}+\frac{4y^2}{(x^2+y^2)^2}=\frac{4}{x^2+y^2}.$$

This looks quite similar to our function, and it's easy to see that f should be written in terms of u and v as $((u^2+v^2)/4)^2 = (u^2+v^2)/16$. We also get $x = 2u/(u^2+v^2)$ and $y = 2v/(u^2+v^2)$. This allows us to compute the Jacobian as

$$\begin{aligned} J &= |\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}| \\ &= |\begin{vmatrix} \frac{2(u^2+v^2)-2u(2u)}{(u^2+v^2)^2} & -\frac{4uv}{(u^2+v^2)^2} \\ -\frac{4uv}{u^2+v^2} & \frac{2(u^2+v^2)-2v(2v)}{(u^2+v^2)^2} \end{vmatrix}| \\ &= \cdots = \frac{4}{(u^2+v^2)^2}. \end{aligned}$$

So the integral is

$$\int_{u=1/3}^{1} \int_{v=1/4}^{1} \frac{(u^2+v^2)^2}{16} \frac{4}{(u^2+v^2)^2} \, dv \, du = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{8}.$$

6. *Integrate $f(x, y) = x$ over a circle of radius 1 centered at $(3, 0)$.*

7. *Find the center of mass of a right half-disk of radius a centered at $(0, 0)$, using the coordinate system of your choice.*

   We'll do this in polar coordinates. The area of the region is $\pi/2$ since it's a half-circle. The y coordinate of the center of mass is obviously 0 by symmetricy. The x coordinate is the intgegral of the function x with respect to area, which in polar coordinates is expressed as $r\cos\theta$. Thus our answer is

$$\bar{x} = \frac{2}{\pi} \int_{r=0}^{1} \int_{\theta=-\pi/2}^{\pi/2} r\cos\theta r \, dr \, d\theta = \frac{2}{\pi} \int_0^1 2r^2 \, dr = \frac{2}{\pi}\frac{2}{3} = \frac{4}{3\pi}.$$

   So the center of mass is $(4/(3\pi), 0)$.