18.089 REVIEW OF MATHEMATICS

HOMEWORK 3, DUE ON FRIDAY, JULY 5

Solve as many problems as you want. Only problems labeled with a $\star$ are required.

Thursday, July 11.

Exercise 1 ($\star$)**.** Verify the normal form of Green's theorem $\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA$ by computing both sides in the following examples.

- C a square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, oriented counterclockwise, and $\vec{F} = (x^2)\hat{\imath} + (y^2)\hat{\jmath}$.
- C the unit circle, and $\vec{F} = (x^2 y)\hat{\imath} + (xy^2)\hat{\jmath}$.

Exercise 2 ($\star$)**.** For what a, b, c will the field $(ay^2 z)\hat{\imath} + yz(bx + z)\hat{\jmath} + (y^2(x + cz) - 3z^2)\hat{k}$ be conservative? For this value of a, b, c, compute $\int_C \vec{F} \cdot d\vec{r}$, where C is a straight line from $(0,0,0)$ to $(1,1,1)$. (You can either integrate directly or find a potential function.)

Friday, July 12.

Exercise 3 ($\star$)**.** Let $\vec{F} = (2xyz)\hat{\imath} + (x^2 z + z)\hat{\jmath} + (x^2 y + y + 1)\hat{k}$. Is $\vec{F}$ conservative? (yes, but check it) Find a function $f(x,y,z)$ with $\nabla f = \vec{F}$.

Exercise 4 ($\star$)**.** Compute the fluxes across the indicated surfaces.

- $\vec{F}(x,y) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, S is a disk of radius 2 parallel to the xy-plane, at height 1.
- $\vec{F}(x,y) = x^2\hat{\imath} + y^2\hat{\jmath}$, S is the side of a cylinder with base $z = 0$, top $z = 3$, and radius 2.
- $\vec{F}(x,y) = x^2\hat{\imath} + y^2\hat{\jmath}$, S is the unit sphere.

Monday, July 15.

Exercise 5 ($\star$)**.** Let S_1 be the part of paraboloid $z = a - x^2 - y^2$ with $z \geq 0$, and let S_2 be the inside of the circle at which S_1 intersects the xy-plane, so that S_1 and S_2 together form a closed surface S.

Check the divergence theorem by computing both sides where S is the surface above and $\vec{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + (z+1)\hat{k}$.

Exercise 6 ($\star$)**.** Let S be the part of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ which lies above the xy plane. Compute the flux of $\vec{F}(x,y,z) = z\hat{k}$ in two ways:

- Directly, by finding $d\vec{S}$ and integrating. You don't actually have to evaluate the integral.
- Indirectly, by computing the flux across a simpler surface and applying the divergence theorem.

Tuesday, July 16.

Exercise 7 ($\star$)**.** Parametrize the following surfaces, and compute the flux of $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ across each one. (Use the formula for flux across a parametrized surface, even if you have a better way.)

- The paraboloid $z = x^2 + y^2$, when $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
- The cone $z^2 = x^2 + y^2$, over the unit circle in the xy-plane.

Exercise 8 ($\star$)**.** Two Stokes questions:

- Check Stokes' theorem by computing both sides, when S is a cylinder of height 2 and radius 3, and $\vec{F} = (-yz)\hat{\imath} + (xz)\hat{\jmath} + (0)\hat{k}$.
- Suppose that $\vec{F}(x, y, z)$ is an everywhere-defined vector field with curl $\vec{F} = 0$. Explain why Stokes' theorem works in this special case.