18.089 REVIEW OF MATHEMATICS

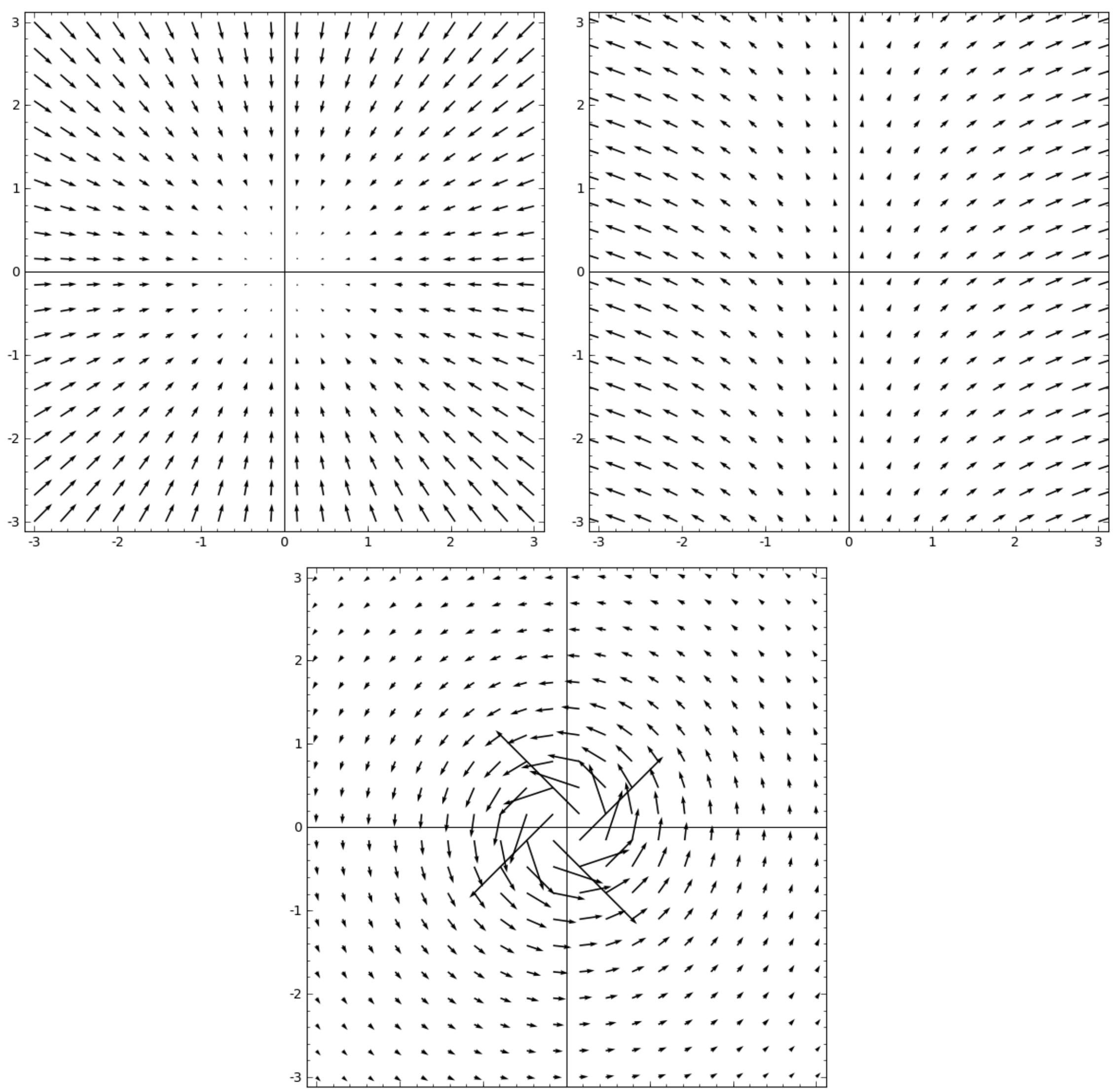
HOMEWORK 3, DUE ON FRIDAY, JULY 5

Solve as many problems as you want. Only problems labeled with a $\star$ are required.

Friday, July 5.

Exercise 1 ($\star$)**.** Sketch the following vector fields:

- $\vec{F}(x,y) = -x\hat{\imath} - y\hat{\jmath}$
- $\vec{F}(x,y) = x\hat{\imath} + \hat{\jmath}$
- $\vec{F}(x,y) = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$



Exercise 2 ($\star$)**.** Let $\vec{F}$ be the vector field $\vec{F}(x,y) = (ay+1)\hat{\imath} + (2x)\hat{\jmath}$. Consider two paths from $(-1,0)$ to $(1,0)$: C_1 a straight line from $(-1,0)$ to $(1,0)$, and C_2 the top half of the unit circle.

Parametrize these two paths, and compute $\int_{C_i} \vec{F} \cdot d\vec{r}$ for both. Give your answer in terms of the parameter a.

[I changed this problem Monday afternoon – if you already did it with the original field, you can leave as is, but the next problem won't be as fun. Sorry!]

For C_1, we want $x(t) = t$ and $y(t) = 0$, with $-1 \le t \le 1$. The integral for this path is

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} M\,dx + N\,dy = \int_{t=-1}^{1} (1)(dt) + (2t)(0)\,dt = 2.$$

This doesn't depend on a. For C_2, we want $x(t) = -\cos t$ and $y(t) = \sin t$, $0 \le t \le \pi$. This gives

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} M\,dx + N\,dy = \int_{t=0}^{\pi} (a\sin t + 1)(\sin t\,dt) + (-2\cos t)(\cos t\,dt)$$
$$= \int_{t=0}^{\pi} a\sin^2 t + \sin t - 2\cos^2 t\,dt = \frac{a\pi}{2} + 2 - \pi = 2 + \frac{(a-2)\pi}{2}.$$

Monday, July 8.

Exercise 3 ($\star$)**.** For some value of a, the vector field from Exercise 2 is conservative. What is this value of a? For that value of a, find the potential function f. Is your answer to exercise 2 consistent with path-independence for this value of a?

To check if it's conservative, we need to see if $M_y = N_x$. We have $M_y = a$ and $N_x = 2$, so this is going to be the case if $a = 2$, so $\vec{F} = (2y+1)\hat{\imath} + (2x)\hat{\jmath}$.

To find the gradient function $f(x,y)$, we need $f_x = 2y + 1$, so $f = 2xy + x + g(y)$. Then $f_y = 2x + g'(y)$, which is supposed to be $2x$. So $g'(y) = 0$ and g is just a constant function, which we can take to be 0. So $f = 2xy + x$.

Observe that for $a = 2$ the values of the two line integrals are both 2, so our answer is indeed consistent with path-independence. We could also get the answer in this case using the fundamental theorem: $f(1,0) - f(-1,0) = 1 - (-1) = 2$.

Exercise 4. Which of the following regions are simply connected? Connected?

- The upper half of the plane.
- The plane, with the square $-1 \le x \le 1$, $-1 \le y \le 1$ deleted.
- The region where $\vec{F} = \frac{1}{y}\hat{\imath} + \hat{\jmath}$ is defined.

The upper half of the plane is both. The plane with the square deleted is connected (one piece), but isn't simply connected. The vector field in question is defined except when $y = 0$. This region has two pieces, so it isn't connected, but there are no holes, so it is simply connected.

Exercise 5 ($\star$)**.** Let C be a curve which follows the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$ and then follows a horizontal line back from $(1,1)$ to $(-1,1)$. Compute

$$\oint_C y^2\,dx + 4xy\,dy$$

using the definition of a line integral.

First along the parabola: $x(t) = t$, $y(t) = t^2$. We get

$$\int_{C_1} y^2\,dx + 4xy\,dy = \int_{t=-1}^{1} (t^4)(dt) + (4t^3)(2t\,dt) = \int_{t=-1}^{1} 9t^4\,dt = \frac{18}{5}.$$

For the line: $x(t) = -t$, $y(t) = 1$, $-1 \leq t \leq 1$.

$$\int_{C_2} y^2\,dx + 4xy\,dy = \int_{t=-1}^{1} (1)(-dt) + (-4t)(0\,dt) = -2 + 0 = -2.$$

So

$$\oint_C y^2\,dx + 4xy\,dy = \frac{18}{5} - 2 = \frac{8}{5}.$$

Tuesday, July 9.

Exercise 6 ($\star$)**.** Compute the integral from Exercise 5 again, this time using Green's theorem.

It should be

$$\int_{x=-1}^{1} \int_{y=x^2}^{1} (4y - 2y)\,dy\,dx = \int_{x=-1}^{1} (1 - x^4)\,dx = \int_{x=-1}^{1} = \frac{8}{5}.$$

This matches our earlier answer.

Exercise 7 ($\star$)**.** Consider the line integral

$$\oint_C (x^2y + x^3 - x)\,dx + (4x - 2y^2x + e^y)\,dy,$$

where C is a simple closed curve. [Corrected Tues – sorry!]

(1) Use Green's theorem to convert this to a double integral over the region R enclosed by C.

Green's theorem says that

$$\oint_C (x^2y + x^3 - x)\,dx + (4x - 2y^2x + e^y)\,dy = \iint_R (4 - 2y^2 - x^2).$$

(2) For what closed curve C, going counterclockwise, is this integral as large as possible?

To make this integral large, we want C to enclose the entire region where curl $\vec{F} = 4 - 2y^2 - x^2$ is positive, and none of the area where it's negative. If R contains points where this is negative it will decrease the double integral, whereas if R leaves out some points where it's positive, we're missing an chance to make the integral larger. The region where this is positive is the inside of the ellipse $x^2 + 2y^2 = 4$, and so our curve C should be one loop around this ellipse.

Wednesday, July 10.

Exercise 8 ($\star$)**.** Let $\vec{F}$ be the vector field

$$\vec{F}(x,y) = 2\left(\frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}\right) + 5\left(\frac{-y}{(x-3)^2+y^2}\hat{\imath} + \frac{(x-3)}{(x-3)^2+y^2}\hat{\jmath}\right).$$

Let C be a circle of radius 100 centered at the origin, going counterclockwise. What is

$$\oint_C \vec{F} \cdot d\vec{r}.$$

Let's write $\vec{F} = 2\vec{F}_1 + 5\vec{F}_2$, where $\vec{F}_1$

$$\vec{F}_1 = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}, \qquad \vec{F}_2 = \frac{-y}{(x-3)^2+y^2}\hat{\imath} + \frac{(x-3)}{(x-3)^2+y^2}\hat{\jmath}.$$

The field $\vec{F}_1$ is defined except at $(0,0)$, and $\vec{F}_2$ is defined except at $(3,0)$. Define C_1 and C_2 to be circles of radius 1 centered at these two points. According to Green's theorem for non-simply connected regions,

$$\oint_C \vec{F}\cdot d\vec{r} + \oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl} \vec{F}\,dx\,dy$$

We want to know $\oint_C \vec{F}\cdot d\vec{r}$, so we're going to have to figure out the other three terms. We know that $\operatorname{curl}\vec{F}_1 = 0$ (I worked this out in class). $\vec{F}_2$ is the same thing as $\vec{F}_1$, but shifted to the right by three units. So $\operatorname{curl}\vec{F}_2 = 0$ as well. This means that $\operatorname{curl}\vec{F} = 2\operatorname{curl}\vec{F}_1 + 5\operatorname{curl}\vec{F}_2 = 0$, and so $\iint_R \operatorname{curl}\vec{F} = 0$. To compute $\oint_{C_1} \vec{F}\cdot d\vec{r}$, split it up as

$$2\oint_{C_1} \vec{F}_1\cdot d\vec{r} + 5\oint_{C_1} \vec{F}_2\cdot d\vec{r}.$$

Now, $\oint_{C_1} \vec{F}_1\cdot d\vec{r}$ is something we computed in class; it's -2π (the minus sign comes since we're going clockwise instead of counterclockwise). Green's theorem doesn't apply here, because $\vec{F}_1$ isn't defined everywhere inside C_1. For $\oint_{C_1} \vec{F}_2\cdot d\vec{r}$, Green's theorem does come into play: the field $\vec{F}_2$ is undefined only at $(3,0)$, so there's no problem. The curl of $\vec{F}_2$ is 0, so Green's theorem gives this integral as 0. This means

$$\oint_{C_1} \vec{F}\cdot d\vec{r} = 2\oint_{C_1} \vec{F}_1\cdot d\vec{r} + 5\oint_{C_1} \vec{F}_2\cdot d\vec{r} = 2(2\pi) + 5(0) = 4\pi.$$

$\oint_{C_2} \vec{F}\cdot d\vec{r}$ works the same way. We get $\oint_{C_2} \vec{F}_1\cdot d\vec{r} = 0$ since Green's theorem is valid here, and we get $\oint_{C_2} \vec{F}_2\cdot d\vec{r} = 2\pi$, since this is the same integral as before, just shifted to the right by three. So

$$\oint_{C_2} \vec{F}\cdot d\vec{r} = 2\oint_{C_1} \vec{F}_1\cdot d\vec{r} + 5\oint_{C_1} \vec{F}_2\cdot d\vec{r} = 2(0) + 5(-2\pi) = -10\pi.$$

All together, we now have

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} + \oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r} &= \iint_R \operatorname{curl} \vec{F}\,dx\,dy \\ \oint_C \vec{F}\cdot d\vec{r} + (-4\pi) + (-10\pi) &= 0 \\ \oint_C \vec{F}\cdot d\vec{r} &= 14\pi.\end{aligned}$$

Thursday, July 11.

We'll have a pset due next Friday, but problems only through Tuesday to avoid overlap with the test (at least the first part of which I'll release Wednesday).