18.089 REVIEW OF MATHEMATICS

HOMEWORK 3, DUE ON FRIDAY, JULY 5

Solve as many problems as you want. Only problems labeled with a $\star$ are required.

Friday, July 5.

Exercise 1 ($\star$)**.** Sketch the following vector fields:

- $\vec{F}(x,y) = -x\hat{\imath} - y\hat{\jmath}$
- $\vec{F}(x,y) = x\hat{\imath} + \hat{\jmath}$
- $\vec{F}(x,y) = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$

Exercise 2 ($\star$)**.** Let $\vec{F}$ be the vector field $\vec{F}(x,y) = (ay+1)\hat{\imath} + (2x)\hat{\jmath}$. Consider two paths from $(-1,0)$ to $(1,0)$: C_1 a straight line from $(-1,0)$ to $(1,0)$, and C_2 the top half of the unit circle.

Parametrize these two paths, and compute $\int_{C_i} \vec{F} \cdot d\vec{r}$ for both. Give your answer in terms of the parameter a.

[I changed this problem Monday afternoon – if you already did it with the original field, you can leave as is, but the next problem won't be as fun. Sorry!]

Monday, July 8.

Exercise 3 ($\star$)**.** For some value of a, the vector field from Exercise 2 is conservative. What is this value of a? For that value of a, find the potential function f. Is your answer to exercise 2 consistent with path-independence for this value of a?

Exercise 4. Which of the following regions are simply connected? Connected?

- The upper half of the plane.
- The plane, with the square $-1 \le x \le 1$, $-1 \le y \le 1$ deleted.
- The region where $\vec{F} = \frac{1}{y}\hat{\imath} + \hat{\jmath}$ is defined.

Exercise 5 ($\star$)**.** Let C be a curve which follows the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$ and then follows a horizontal line back from $(1,1)$ to $(-1,1)$. Compute

$$\oint_C y^2\,dx + 4xy\,dy$$

using the definition of a line integral.

Tuesday, July 9.

Exercise 6 ($\star$)**.** Compute the integral from Exercise 5 again, this time using Green's theorem.

Exercise 7 ($\star$)**.** Consider the line integral

$$\oint_C (x^2y + x^3 - x)\,dx + (4x - 2y^2x + e^y)\,dy,$$

where C is a simple closed curve. [Corrected Tues – sorry!]

(1) Use Green's theorem to convert this to a double integral over the region R enclosed by C.
(2) For what closed curve C, going counterclockwise, is this integral as large as possible?

Wednesday, July 10.

Exercise 8 ($\star$)**.** Let $\vec{F}$ be the vector field

$$\vec{F}(x,y) = 2\left(\frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}\right) + 5\left(\frac{-y}{(x-3)^2+y^2}\hat{\imath} + \frac{(x-3)}{(x-3)^2+y^2}\hat{\jmath}\right).$$

Let C be a circle of radius 100 centered at the origin, going counterclockwise. What is

$$\oint_C \vec{F}\cdot d\vec{r}.$$

[Hint: use Green's theorem for non-simply connected regions. What are the two points where $\vec{F}$ isn't defined?]

Maybe this wasn't a fair problem, so here's a bit more of a hint. Let's write $\vec{F} = 2\vec{F}_1 + 5\vec{F}_2$, where $\vec{F}_1$ is the thing on the left and $\vec{F}_2$ is the thing on the right. You're going to want to define C_1 and C_2 to be circles of radius 1 centered at the points where $\vec{F}_1$ and $\vec{F}_2$ respectively aren't defined (clockwise or counterclockwise?). According to Green's theorem,

$$\oint_C \vec{F}\cdot d\vec{r} + \oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r} = \iint_R \text{curl } \vec{F}\, dx\, dy$$

We want to know $\oint_C \vec{F}\cdot d\vec{r}$, so we're going to have to figure out the other three terms. The double integral won't be too hard once you find the curl of $\vec{F}$. For $\oint_{C_1} \vec{F}\cdot d\vec{r}$, split it up as

$$2\oint_{C_1} \vec{F}_1\cdot d\vec{r} + 5\oint_{C_1} \vec{F}_2\cdot d\vec{r}.$$

For one of these, Green's theorem doesn't apply because the field in question isn't defined on the inside of C_1. So you'll have to compute a line integral directly. For the other one, Green's theorem does work because the field is defined inside C_1, so you can find the integral that way. Then apply the same strategy for $\oint_{C_2} \vec{F}\cdot d\vec{r}$ and put it all together.

Thursday, July 11.

We'll have a pset due next Friday, but problems only through Tuesday to avoid overlap with the test (at least the first part of which I'll release Wednesday).