18.089 HOMEWORK 0 SOLUTION

Exercise 1. Compute the derivatives of the following functions:
(a) $y(x) = x^3$ (using the definition).
(b) $y(x) = x^n$ (using the definition).
(c) $y(x) = 1/\sqrt{x}$.
(d) $y(x) = 1/\sqrt{x^2-1}$.
(e) $y(x) = (x^3+1)(x^2-2x)$.
(f) $y(x) = f(x)g(x)h(x)$.
(g) $y(x) = \sqrt{x}/(x^2-1)$.
(h) $y(x) = \left(1/(x^2+1)\right)^2$.
(i) $y(x) = (x^4+3x)^3$.

Solution. (a)

$$y'(x) = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2.$$

(b) Using binomial theorem,

$$y'(x) = \lim_{h\to 0}\frac{(x+h)^n-x^n}{h} = \lim_{h\to 0}\frac{1}{h}\sum_{k=1}^{n}\binom{n}{k}h^k x^{n-k} = \sum_{k=1}^{n}\lim_{h\to 0}\binom{n}{k}h^{k-1}x^{n-k} = nx^{n-1}.$$

(c) Rewrite $y(x) = x^{-1/2}$ and use power rule to get $y'(x) = -x^{-3/2}/2$.
(d) Rewrite $y(x) = (x^2-1)^{-1/2}$ and use chain rule:

$$y'(x) = -\frac{1}{2}(x^2-1)^{-3/2}\cdot(2x) = \frac{x}{(x^2-1)^{3/2}}.$$

(e) Expand the polynomial $y(x) = x^5-2x^4+x^2-2x$ and differentiate term-by-term to get $y'(x) = 5x^4 - 8x^3+2x-2$.
(f) Treat $g(x)h(x)$ as a single function and apply product rule

$$y'(x) = f'(x)g(x)h(x)+f(x)(gh)'(x)$$

Apply product rule again to calculate $(gh)'$:

$$y'(x) = f'(x)g(x)h(x)+f(x)\left[g'(x)h(x)+g(x)h'(x)\right] = f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x).$$

(g) Use quotient rule:

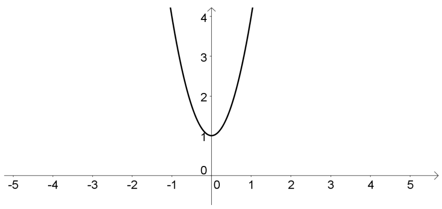
$$y'(x) = \frac{(x^{-1/2}/2)(x^2-1)-x^{1/2}(2x)}{(x^2-1)^2} = \frac{-3x^{3/2}/2-x^{-1/2}/2}{(x^2-1)^2}.$$

(h) Rewrite $y(x) = (x^2+1)^{-2}$ and use chain rule $y'(x) = -2(x^2+1)^{-3}(2x) = -4x(x^2+1)^{-3}$.
(i) Use chain rule: $y'(x) = 3(4x^3+3)(x^4+3x)^2$.

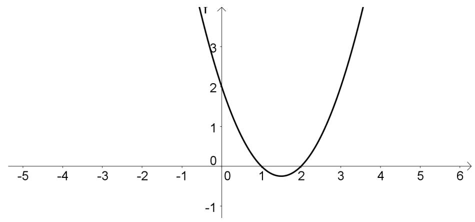
Exercise 2. Study the following functions (i.e. graph them and determine the nature of critical points):
(a) $y = 3x^2+1$.
(b) $y = x^2-3x+2$.
(c) $y = x^3$.
(d) $y = x^3+1$.

Date: June 11, 2013.

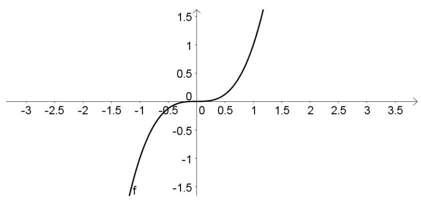
Solution. Graphs:



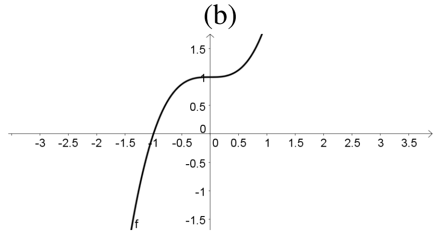
(a)



(b)



(c)



(d)

Critical points:

(a) $y' = 6x$. Solving $y' = 0$ gives $x = 0$, so critical point is $(0,1)$. $y'' = 6 > 0$ at $x = 0$, so it is a local minimum.

(b) $y' = 2x - 3$. Solving $y' = 0$ gives $x = 3/2$, so critical point is $(3/2, -1/4)$. $y'' = 2 > 0$ at $x = 3/2$, so it is a local minimum.

(c) $y' = 3x^2$. Solving $y' = 0$ gives $x = 0$, so critical point is $(0,0)$. $y' > 0$ for both $x < 0$ and $x > 0$, so it is a saddle point.

(d) $y' = 3x^2$. Solving $y' = 0$ gives $x = 0$, so critical point is $(0,1)$. $y' > 0$ for both $x < 0$ and $x > 0$, so it is a saddle point.

Exercise 3. Compute the tangent lines of the above functions at the points $(0,1)$, $(1,0)$, $(0,0)$ and $(1,2)$ respectively.

Solution. (a) At $(0,1)$, $y' = 0$, so the tangent line is $y - 1 = (0)(x - 0)$, i.e., $y = 1$.

(b) At $(1,0)$, $y' = -1$, so the tangent line is $y - 0 = (-1)(x - 1)$, i.e., $y = -x + 1$.

(c) At $(0,0)$, $y' = 0$, so the tangent line is $y - 0 = (0)(x - 0)$, i.e., $y = 0$.

(d) At $(1,2)$, $y' = 3$, so the tangent line is $y - 2 = (3)(x - 1)$, i.e., $y = 3x - 1$.

Exercise 4. Compute the tangent line of the function $y = \sqrt{x^3 + 3x}$ at the point $(1,2)$.

Solution.

$$\frac{dy}{dx} = \frac{1}{2}(x^3 + 3x)^{-1/2}(3x^2 + 3) = \frac{3}{2}(x^2 + 1)(x^3 + 3x)^{-1/2}.$$

At $x = 1$, $dy/dx = 3/2$, so the tangent line is $y - 2 = (3/2)(x - 1)$, i.e., $y = 1.5x + 0.5$.

Exercise 5. Compute the tangent line of the curve $x^2 + y^2 = 25$ at the points $(3,4)$ and $(4,3)$. Where do these lines intersect?

Solution. Differentiating $x^2 + y^2 = 25$ gives $2x + 2yy' = 0$, i.e., $y' = -x/y$ if $y \neq 0$. Hence, at $(3,4)$, $y' = -3/4$ and the tangent line is

$$y = -\frac{3}{4}x + \frac{25}{4}.$$

At $(4,3)$, $y' = -4/3$ and the tangent line is

$$y = -\frac{4}{3}x + \frac{25}{3}.$$

The two lines intersect at $(25/7, 25/7)$.

Exercise 6. Minimize the perimeter of a rectangle with area 20.

Solution. If l is the length of the rectangle, then $20/l$ is its width, so its perimeter is

$$P = 2\left(l + \frac{20}{l}\right) = 2l + \frac{40}{l}.$$

Note that $dP/dl = 2 - 40/l^2$, so solving $dP/dl = 0$ gives $l = \sqrt{20}$.

As $l \to 0$, $P \to \infty$.

At $l = \sqrt{20}$, $P = 8\sqrt{5}$.

As $l \to \infty$, $P \to \infty$.

Hence the minimum is achieved at $l = \sqrt{20}$ (i.e., when the rectangle is a sqaure) and the minimum perimeter is $8\sqrt{5}$.

Exercise 7. A ball travels on the parabola $y - x^2 = 0$. At each time t, denote by $x(t)$ and $y(t)$ the projections of the ball on the x and y-axis respectively. If you know that the speed of $x(t)$ is constant and equal to 3, what is the speed of $y(t)$ when $x(t) = 1$ (and hence $y(t) = 1$)?

Solution. Since $dy/dx = 2x$, by chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (2x)(3) = 6x,$$

i.e., $y'(t) = 6$ when $x(t) = 1$.

Exercise 8. Use trigonometric formulas and implicit differentiation to show that $(\cos^{-1})' = -1/\sqrt{1-x^2}$.

Solution. Let $y = \cos^{-1} x$. Then $x = \cos y$. Differentiating both sides with respect to x,

$$1 = -y' \sin y.$$

Note that $\sin^2 y = 1 - \cos^2 y = 1 - x^2$. Also, since $0 \le y \le \pi$, $\sin y \ge 0$, so

$$y' = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-x^2}}.$$

Exercise 9. Compute the following integrals:

(a) $\int \left[x^3/(x^4+5)\right] dx$.

(b) $\int \theta \sin(\theta^2) d\theta$.

(c) $\int x e^{x^2} dx$.

(d) $\int \left(3x/\sqrt{x^2+1}\right) dx$.

Solution. (a)

$$\int \frac{x^3 dx}{x^4+5} = \frac{1}{4}\int \frac{d(x^4+5)}{x^4+5} = \frac{1}{4}\log(x^4+5) + C.$$

(b)

$$\int \theta \sin(\theta^2) d\theta = \frac{1}{2}\int \sin(\theta^2) d(\theta^2) = -\frac{1}{2}\cos(\theta^2) + C.$$

(c)

$$\int x e^{x^2} dx = \frac{1}{2}\int e^{x^2} d(x^2) = \frac{1}{2}e^{x^2} + C.$$

(d)

$$\int \frac{3x dx}{\sqrt{x^2+1}} = \frac{3}{2}\int \frac{d(x^2+1)}{\sqrt{x^2+1}} = 3\sqrt{x^2+1} + C.$$

Exercise 10. Compute the following integrals using substitutions:

(a) $\int e^x(e^x+1)^b dx$, $u = e^x+1$.

(b) $\int dx/\sqrt{9-4x^2} = (1/3)\int dx/\sqrt{1-(2x/3)^2}$.

(c) $\int \left[(2x+1)/(x^2+x+1)\right] dx$.

(d) $\int \left(x/\sqrt{x^2+1}\right) dx$.

(e) $\int \left(\cos x/\sqrt{1+\sin x}\right) dx$.

Solution. (a) For $u = e^x+1$, $du = e^x dx$, so if $b \neq -1$, then

$$\int e^x(e^x+1)^b dx = \int u^b du = \frac{u^{b+1}}{b+1} + C = \frac{(e^x+1)^{b+1}}{b+1} + C.$$

If $b=-1$, then

$$\int e^x(e^x+1)^{-1} dx = \int u^{-1} du = \log u + C = \log(e^x+1) + C.$$

(b) For $u = 2x/3$, $du = (2/3)dx$, so

$$\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{1-(2x/3)^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\cos^{-1} u + C = \frac{1}{2}\cos^{-1}\left(\frac{2x}{3}\right) + C.$$

(c) For $u = x^2+x+1$, $du = (2x+1)dx$, so

$$\int \frac{(2x+1)dx}{x^2+x+1} = \int \frac{du}{u} = \log u + C = \log(x^2+x+1) + C.$$

(d) For $u = x^2+1$, $du = 2xdx$, so

$$\int \frac{xdx}{\sqrt{x^2+1}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+1} + C.$$

(e) For $u = 1+\sin x$, $du = \cos x dx$, so

$$\int \frac{\cos x dx}{\sqrt{1+\sin x}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1+\sin x} + C.$$

Exercise 11. Compute the following integral: $\int dx/(x^2+6x+25)$.

Solution. Note that $x^2+6x+25 = (x+3)^2+4^2$, so if $x+3 = 4\tan u$, then $dx = 4\sec^2 u du$ and $x^2+6x+25 = 16(\tan^2 u+1) = 16\sec^2 u$. Hence,

$$\int \frac{dx}{x^2+6x+25} = \int \frac{4\sec^2 u du}{16\sec^2 u} = \frac{1}{4}\int du = \frac{u}{4} + C = \frac{1}{4}\tan^{-1}\left(\frac{x+3}{4}\right) + C.$$

Exercise 12. Find the area between the curves $y = x^2+2$ and $y = 4-x^2$.

Solution. The two curves intersect at $x = \pm 1$, so the area is

$$\int_{-1}^{1} \left[(4-x^2)-(x^2+2)\right] dx = \int_{-1}^{1} (2-2x^2) dx = \left[2x - \frac{2}{3}x^3\right]\Big|_{-1}^{1} = \frac{8}{3}.$$

Exercise 13. Find the volume of the solid obtained by revolving around the x-axis the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 4$.

Solution. Cut the solid into disks of thickness dx. Each disk has radius y, so the volume of a disk is $\pi y^2 dx = \pi x dx$. Hence the total volume of the solid is

$$V = \pi \int_0^4 x dx = \frac{\pi}{2} x^2\Big|_0^4 = 8\pi.$$

Exercise 14. Calculate the following integrals:

(a) $\int (x+2)dx/\left[(x-3)^2(x+1)\right]$.

(b) $\int (x+2)dx/[(x-6)(x+5)]$.

(c) $\int x^3dx/[(x-2)(x+2)]$ (note that this is not a proper fraction).

Solution. (a) Write

$$\frac{x+2}{(x-3)^2(x+1)} = \frac{5/4}{(x-3)^2} - \frac{1/16}{x-3} + \frac{1/16}{x+1}.$$

Hence,

$$\int \frac{(x+2)dx}{(x-3)^2(x+1)} = \frac{5}{4}\int \frac{dx}{(x-3)^2} - \frac{1}{16}\int \frac{dx}{x-3} + \frac{1}{16}\int \frac{dx}{x+1} = -\frac{5}{4(x-3)} + \frac{1}{16}\log\left|\frac{x+1}{x-3}\right| + C.$$

(b) Write

$$\frac{x+2}{(x-6)(x+5)} = \frac{8/11}{x-6} + \frac{3/11}{x+5}.$$

Hence,

$$\int \frac{(x+2)dx}{(x-6)(x+5)} = \frac{8}{11}\int \frac{dx}{x-6} + \frac{3}{11}\int \frac{dx}{x+5} = \frac{8}{11}\log|x-6| + \frac{3}{11}\log|x+5| + C.$$

(c) Write

$$\frac{x^3}{(x-2)(x+2)} = x + \frac{4x}{x^2-4}.$$

Hence,

$$\int \frac{x^3dx}{(x-2)(x+2)} = \int xdx + \int \frac{4xdx}{x^2-4} = \int xdx + 2\int \frac{d(x^2-4)}{x^2-4} = \frac{1}{2}x^2 + 2\log|x^2-4| + C.$$

Exercise 15. Compute

(a) $\int_1^{\sqrt{e}} x^3 \log 2xdx$.

(b) $\int e^{3x}\cos 2xdx$.

(c) $\int (x^2+1)\cos 3xdx$.

Solution. (a)

$$\begin{aligned}\int_1^{\sqrt{e}} x^3\log 2xdx &= \frac{1}{4}\int_1^{\sqrt{e}} \log 2xd(x^4)\\ &= \frac{1}{4}(e^2-1)\log 2 + \frac{1}{8}e^2 - \frac{1}{4}\int_1^{\sqrt{e}} x^4 d(\log 2x)\\ &= \frac{1}{4}(e^2-1)\log 2 + \frac{1}{8}e^2 - \frac{1}{4}\int_1^{\sqrt{e}} x^3dx\\ &= \frac{1}{4}(e^2-1)\log 2 + \frac{1}{16}(e^2+1).\end{aligned}$$

(b)

$$\begin{aligned}\int e^{3x}\cos 2xdx &= \frac{1}{2}\int e^{3x}d\sin 2x\\ &= \frac{1}{2}e^{3x}\sin 2x - \frac{1}{2}\int \sin 2xde^{3x}\\ &= \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}\int e^{3x}d\cos 2x\\ &= \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x - \frac{9}{4}\int e^{3x}\cos 2xdx.\end{aligned}$$

Solving for $\int e^{3x}\cos 2x dx$, we get

$$\int e^{3x}\cos 2x dx = \frac{2}{13}e^{3x}\sin 2x + \frac{3}{13}e^{3x}\cos 2x + C.$$

Alternatively, recall $2\cos\theta = e^{i\theta} + e^{-i\theta}$, so

$$\begin{aligned}\int e^{3x}\cos 2x dx &= \frac{1}{2}\int e^{3x}(e^{2ix} + e^{-2ix})dx \\ &= \frac{1}{2}\int (e^{(3+2i)x} + e^{(3-2i)x})dx \\ &= \frac{1}{2}\left(\frac{e^{(3+2i)x}}{3+2i} + \frac{e^{(3-2i)x}}{3-2i}\right) + C \\ &= \frac{1}{2}e^{3x}\cdot\frac{(3-2i)e^{2ix} + (3+2i)e^{-2ix}}{13} + C \\ &= \frac{2}{13}e^{3x}\sin 2x + \frac{3}{13}e^{3x}\cos 2x + C.\end{aligned}$$

(c)

$$\begin{aligned}\int (x^2+1)\cos 3x dx &= \frac{1}{3}\int (x^2+1) d\sin 3x \\ &= \frac{1}{3}(x^2+1)\sin 3x - \frac{1}{3}\int \sin 3x d(x^2+1) \\ &= \frac{1}{3}(x^2+1)\sin 3x + \frac{2}{9}\int x d\cos 3x \\ &= \frac{1}{3}(x^2+1)\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{9}\int \cos 3x dx \\ &= \frac{1}{3}(x^2+1)\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C \\ &= \frac{1}{27}(9x^2+7)\sin 3x + \frac{2}{9}x\cos 3x + C.\end{aligned}$$

Exercise 16. Compute the area inside the circle $x^2 + y^2 = 4$ and the parabola $y = x^2$.

Solution. Suppose the two curves intersect at $x = \pm\alpha$. Then, we have $\alpha^2 + \alpha^4 = 4$. Note that the desired area is given by

$$A = \int_{-\alpha}^{\alpha}\left(\sqrt{4-x^2} - x^2\right)dx.$$

By the substitution $x = 2\sin\theta$,

$$A = \int_{-\beta}^{\beta}(2\cos\theta - 4\sin^2\theta)(2\cos\theta)d\theta = 4\beta + 2\sin 2\beta - \frac{16}{3}\sin^3\beta,$$

where $\alpha = 2\sin\beta$. Hence,

$$\cos\beta = \sqrt{1 - \left(\frac{\alpha}{2}\right)^2} = \frac{1}{2}\sqrt{4-\alpha^2} = \frac{1}{2}\sqrt{\alpha^4} = \frac{\alpha^2}{2}.$$

Therefore, we have

$$A = 4\beta + \alpha^3 - \frac{2}{3}\alpha^3 = 4\beta + \frac{1}{3}\alpha^3 = 4\sin^{-1}\left(\frac{\alpha}{2}\right) + \frac{1}{3}\alpha^3.$$

Solving $\alpha^2 + \alpha^4 = 4$ gives $\alpha = \sqrt{(\sqrt{17}-1)/2}$, so

$$A = 4\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{\sqrt{17}-1}{2}}\right) + \frac{1}{3}\left(\frac{\sqrt{17}-1}{2}\right)^{3/2} \approx 3.35001.$$