18.089 Exam 3

Due July 22, 2013

Name:

This exam consists of eight problems, not arranged in any particular order. Please solve all problems in the space provided (or attaching additional sheets as necessary), showing all work as neatly and cleanly as possible.

All work must be your own. In particular, you may not seek help from other students in the class or any "live" internet resources. But feel free to use reference works such as your class notes, a textbook, or Wikipedia. If you have any questions about what constitutes an acceptable source, please ask.

Problem	**Value**	**Score**
Problem 1	10	
Problem 2	20	
Problem 3	20	
Problem 4	20	
Problem 5	20	
Problem 6	20	
Problem 7	20	
Problem 8	20	
Total	150	

Problem 1 (3+3+3+1 free)**.** True/false: give a brief explanation.

1. (T/F) If $\vec{F}$ is any vector field, then $\nabla \times (\nabla \times \vec{F}) = 0$.

2. (T/F) The fundamental theorem of calculus for line integrals could be used to evaluate $\int_C (e^x + y^2)\,dx + (2xy)\,dy$, where C is goes from $(1,0)$ to $(0,1)$ along a circle.

3. (T/F) Stokes' theorem could be used to replace $\iint_S (x\hat{\imath}+y\hat{\jmath}+z\hat{k})\cdot\hat{n}\,dA$ by an appropriate line integral around C, where S is the top half of the unit sphere, and C is a circle in the xy-plane.

Problem 2 (10+10)**.** Let $\vec{F}(x,y) = (x^2)\hat{\imath} + (2x+y)\hat{\jmath}$. Take C_1 to be a straight line from $(0,1)$ to $(6,9)$, and C_2 to follow a circular path between these points, centered at the midpoint $(3,5)$, and passing below C_1.

1. Compute $\int_{C_1} \vec{F} \cdot d\vec{r}$.

2. What is $\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$?

Problem 3 (10+10)**.** Find the areas of the following regions. You can use any method you want.

1. The region bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $x + y = 1$, and $x + y = 2$.

2. The region bounded by the two axes and the curve $x(t) = \cos^2 t$, $y(t) = \sin^3 t$ (you can use a computer for help with integrals).

Problem 4 (10+10)**.** Consider the surface defined by the implicit equation $z = (x^2+y^2+z^2)^2$.

1. What is the equation for this surface in spherical coordinates?

2. Set up integrals to find the z-coordinate of the center of mass of this region.

Problem 5 (10+10)**.** This problem deals with the vector field $\vec{F} = x\hat{\imath} + xy\hat{\jmath} + z^2\hat{k}$. Let S be a cylinder of radius 3 and height 2, with base in the xy-plane, including both the top and the bottom.

1. Compute $\oiint_S \vec{F} \cdot \hat{n}\, dA$ using your favorite method.

2. Use Stokes' theorem to compute $\iint_{S_{\text{side}}} \operatorname{curl} \vec{F} \cdot \hat{n}\, dA$ (don't include the top and the bottom!)

Problem 6 (10+10)**.** Let $\vec{F} = (-6x + ay)\hat{\imath} + (-6y + 2x)\hat{\jmath}$.

1. For what value of a is this a conservative field? For that value, find a potential function.

2. What are the possible values of $\int_C \vec{F} \cdot d\vec{r}$, where C is allowed to be any curve (not necessarily closed) which is contained in the unit disk? (assume the value of a from the first part)

Problem 7 (10+10). 1. Suppose that $\vec{G}$ is a constant vector field, and that $\vec{F}$ is any vector field. Show that $\nabla \cdot (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \cdot \vec{G}$.

2. Suppose that R is a three-dimensional region, with S its boundary. Let f and g be two functions defined everywhere in R. Explain why

$$\iiint_R (f\Delta g + \nabla f \cdot \nabla g) = \oiint_S f(\nabla g \cdot \hat{n})\, dA.$$

(Remember that Δg is the Laplacian of g, the function $\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}$).

Problem 8. Compute the following flux integrals directly.

1. $\vec{F}(x, y, z) = z\hat{k}$, across the top half of the unit sphere.

2. $\vec{F}(x, y, z) = x\hat{\imath}$ across S, the part of the plane $x = y$ satisfying $0 \leq x \leq 1$, $0 \leq z \leq 2$.