18.089 Exam 2

July 5, 2013

Name:

This exam consists of nine problems, not arranged in any particular order. Please solve all problems in the space provided (or attaching additional sheets as necessary), showing all work as neatly and cleanly as possible.

All work must be your own. In particular, you may not seek help from other students in the class or any "live" internet resources. But feel free to use reference works such as your class notes, a textbook, or Wikipedia. If you have any questions about what constitutes an acceptable source, please ask.

Problem	**Value**	**Score**
Problem 1	15	
Problem 2	20	
Problem 3	30	
Problem 4	15	
Problem 5	30	
Problem 6	20	
Problem 7	20	
Problem 8	25	
Problem 9	25	
Total	200	

Problem 1 (*15 points*)**.** Compute the first five terms (i.e., up to and including the x^4 term) of the Taylor series for $f(x) = e^{2x} \cdot \cos x + \sin(3x)$. (Hint: there are two natural ways to proceed – either by taking derivatives or by using some Taylor series you already know.)

Solution:
Method 1: We mentioned in class that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots,$$
$$\sin x = x - \frac{x^3}{6} + \dots,$$

and

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots.$$

Thus

$$\begin{aligned} e^{2x} &= 1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \frac{(2x)^4}{24} + \dots \\ &= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \dots, \end{aligned}$$

so

$$\begin{aligned} e^{2x}\cos x &= \left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \dots\right) \cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right) \\ &= 1 + 2x + \left(2 - \frac{1}{2}\right)x^2 + \left(\frac{4}{3} - 2 \cdot \frac{1}{2}\right)x^3 + \left(\frac{2}{3} - 2 \cdot \frac{1}{2} + \frac{1}{24}\right)x^4 + \dots \\ &= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{3} - \frac{7x^4}{24} + \dots. \end{aligned}$$

Adding this to

$$\sin(3x) = 3x - \frac{(3x)^3}{6} + \dots = 3x - \frac{9x^3}{2} + \dots$$

yields

$$\boxed{e^{2x}\cos x + \sin(3x) = 1 + 5x + \frac{3x^2}{2} - \frac{25x^3}{6} - \frac{7x^4}{24} + \dots.}$$

Method 2: compute derivatives and apply the general formula. We have

$$\begin{aligned} f'(x) &= 2e^{2x}\cos x - e^{2x}\sin x + 3\cos 3x, \\ f''(x) &= 3e^{2x}\cos x - 4e^{2x}\sin x - 9\sin 3x, \\ f^{(3)}(x) &= 2e^{2x}\cos x - 11e^{2x}\sin x - 27\cos 3x, \end{aligned}$$

and

$$f^{(4)}(x) = -7e^{2x}\cos x - 24e^{2x}\sin x + 81\sin 3x.$$

Thus $f(0) = 1$, $f'(0) = 5$, $f''(0) = 3$, $f^{(3)}(0) = -25$ and $f^{(4)}(0) = -7$, which leads to the same answer as the previous method.

Problem 2 *(10+10 point)*. (a) Determine the angles of the triangle with vertices $P_0 = (1, -4, 1)$, $P_1 = (1, 1, 1)$ and $P_2 = (4, 1, 5)$.

Solution:
$\vec{P_0P_1} = (0, 5, 0)$, $\vec{P_0P_2} = (3, 5, 4)$ and $\vec{P_1P_2} = (3, 0, 4)$.
The angles are given by the formule $\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$, where θ is the angle between $\vec{v}$ and $\vec{w}$.
$\cos(\vec{P_0P_1} \wedge \vec{P_0P_2}) = \frac{25}{5\cdot\sqrt{50}} = \frac{1}{\sqrt{2}}$. Hence $\vec{P_0P_1} \wedge \vec{P_0P_2} = \frac{\pi}{4}$.
$\cos(\vec{P_0P_1} \wedge \vec{P_1P_2}) = 0$. Hence $\vec{P_0P_1} \wedge \vec{P_1P_2} = \frac{\pi}{2}$.
$\cos(\vec{P_0P_2} \wedge \vec{P_1P_2}) = \frac{9+16}{\sqrt{50}\cdot 5} = \frac{1}{\sqrt{2}}$. Hence $\vec{P_0P_2} \wedge \vec{P_1P_2} = \frac{\pi}{4}$.

(b) Determine the implicit equation of the plane containing the above triangle.

Solution:
We compute $\vec{P_0P_1} \times \vec{P_1P_2} = det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 5 & 0 \\ 3 & 0 & 4 \end{pmatrix} = (20, 0, -15)$.
If we take P_0 as the point on the plane, we get the equation $20x - 15z = 5$, which reduces to $4x - 3z = 1$.

Problem 3 (*10+10+10 points*)**.** A particle travels in space so that its position at time t is given by the parametric equation

$$R(t) = (t \sin t, t \cos t, t).$$

(a) Determine the velocity, the acceleration and the speed of the particle at the point $R(0)$.

Solution:
$\vec{v}(t) = (\sin t + t\cos t, \cos t - t \sin t, 1)$.
$|\vec{v}(t)| = \sqrt{2+t^2}$.
$\vec{a}(t) = (2\cos t - t\sin t, -2\sin t - t\cos t, 0)$.
Hence $\vec{v}(0) = (0, 1, 1)$, $|\vec{v}(0)| = \sqrt{2}$ and $\vec{a}(0) = (2, 0, 0)$.

(b) Write down (but do not evaluate) the integral that gives the arclength of the portion of the curve between the points $R(0)$ and $R(\pi/2)$.

solution:
$length = \int_0^{\pi/2} |\vec{v}(t)| dt = \int_0^{\pi/2} \sqrt{2+t^2} dt.$

(c) Show that the curve lies on the surface $x^2 + y^2 - z^2 = 0$ and compute the tangent plane to this surface at $R(\pi/2)$.

solution:
First, let's check that the curve lies on the surface:

$$x(t)^2 + y(t)^2 + z(t)^2 = t^2 \sin^2(t) + t^2 \cos^2(t) - t^2 = t^2 - t^2 = 0$$

so all points on the curve satisfy the equation of the surface.

The point $R(\pi/2)$ has coordinates $(\pi/2, 0, \pi/2)$. Consider $f(x, y, z) = x^2 + y^2 - z^2$. It has gradient $(2x, 2y, -2z)$. At $R(\pi/2)$, this has value $(\pi, 0, -\pi)$. It is a normal vector to the tangent plane to the surface $f(x, y, z) = 0$ at $R(\pi/2)$. Thus the tangent plane has equation:

$$\pi(x - \pi/2) - \pi(z - \pi/2) = 0 \Leftrightarrow x = z.$$

Problem 4 (*10+5 points*)**.** Consider the curve given by the polar equation $r = 1 - 2\cos\theta$.
(a) Compute the area bounded by this curve (be careful with the limits of integration!).

Solution:
Remember that r must always be positive! From the condition $1 - 2\cos\theta \geq 0$ we obtain $\theta \in [\pi/3, 5\pi/3)$. The area is given by

$$\frac{1}{2}\int_{\pi/3}^{5\pi/3}(1-2\cos\theta)^2 d\theta = \frac{1}{2}\int_{\pi/3}^{5\pi/3}(1+4\cos^2\theta - 4\cos\theta) =$$

$$= \frac{2\pi}{3} - 2\left[\sin\theta\right]_{\pi/3}^{5\pi/3} + \frac{1}{2}\int_{\pi/3}^{5\pi/3} 4\cos^2\theta d\theta = \frac{2\pi}{3} + \int_{\pi/3}^{5\pi/3}(1+\cos 2\theta)d\theta =$$

$$= \frac{2\pi}{3} + \frac{4}{3}\pi + \left[\frac{1}{2}\sin 2\theta\right]_{\pi/3}^{5\pi/3} = 2\pi + \frac{3\sqrt{3}}{2}.$$

(b) Write down (but do not evaluate) an integral whose value is the arclength of this curve.

Solution:
Again, you have to be careful with limits of integration! The total length is:

$$length = \int_{\pi/3}^{5\pi/3}\sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta = \int_{\pi/3}^{5\pi/3}\sqrt{1+4\cos^2\theta - 4\cos\theta + 4\sin^2\theta}d\theta =$$

$$= \int_{\pi/3}^{5\pi/3}\sqrt{5-4\cos\theta}d\theta.$$

The arclength function is

$$\int_{\pi/3}^{\theta}\sqrt{5-4\cos\theta}d\theta.$$

(Either answer was acceptable.)

Problem 5 *(10+10+10 points)*. Let $z = f(x, y) = e^{x+y} + e^{xy+1}$.
(a) Find the equation of the tangent plane to the surface at the point $(2, 1, 2e^3)$.

Solution:
$f_x(x, y) = e^{x+y} + ye^{xy+1}$, $f_y(x, y) = e^{x+y} + xe^{xy+1}$. At the point $(2, 1, 2e^3)$ we have $f_x(2, 1) = 2e^3$, $f_y(2, 1) = 3e^3$. The tangent plane is thus given by

$$z = 2e^3 + 2e^3(x - 2) + 3e^3(y - 1)$$

which can be a little simplified, for instance to

$$2e^3x + 3e^3y - z = 5e^3$$

(b) Find the implicit and explicit equation of the line perpendicular to the plane passing through $(2, 1, 2e^3)$.

Solution:
From the equation of the plane we see that the vector $(2e^3, 3e^3, -1)$ is tangent to the plane, hence the equation of the line is

$$\begin{cases} x = 2e^3t + 2 \\ y = 3e^3t + 1 \\ z = -t + 2e^3. \end{cases}$$

If you solve for t you get the implicit equations

$$\begin{cases} y + 3e^3z - 1 - 6e^6 = 0 \\ x + 2e^3z - 2 - 4e^6 = 0. \end{cases}$$

(Note that you need a pair of implicit equations, but that it is not unique.)

(c) Compute the directional derivative at the point $(1, -1)$ in the direction $\vec{v} = (3, -3)$.

Solution:
$\nabla(f)(1, -1) = (f_x(1, -1), f_y(1, -1)) = (0, 2)$, $\vec{v}$ is not a unit vector, hence we have to rescale it and

$$D_{dir\vec{v}}f(1, -1) = \frac{1}{\sqrt{18}}\nabla(f)(1, -1) \cdot \vec{v} = \frac{-6}{3\sqrt{2}} = -\sqrt{2}.$$

Problem 6 *(10+10 points)***.** The plane given by $x+y+2z=2$ and the paraboloid $z=x^2+y^2$ intersect in an ellipse.
(a) Find the points on this ellipse that are nearest to and farthest from the origin.

Solution:

This is a tricky question: it asks us to do a Lagrange-type maximization problem, but where we have two constraints instead of one! There are two routes you can take here: either reduce it to a one-constraint problem, or use the multiple constraint Lagrange method you might have picked up somewhere else (e.g. Wiki).

Let's do the first way: our strategy is going to be to eliminate z as much as possible, and treat this as a maximization problem in x and y. The intersection of the two is where $x+y+2z=2$ and $z=x^2+y^2$, which means that $x+y+2x^2+2y^2-2=0$. Now, the function we want to maximize is $\sqrt{x^2+y^2+z^2}$. As you have probably realized by now, it's always going to be easier to maximize/minimize the square of the distance function (i.e. $x^2+y^2+z^2$), rather than the distance itself. In the intersection, we have

$$f(x,y)=x^2+y^2+z^2=x^2+y^2+\left(\frac{2-x-y}{2}\right)^2.$$

We want to max/min this $f(x,y)$, subject to the constraints $g(x,y)=x^2+y^2+2x^2+2y^2-2=0$. Set up Lagrange: $\nabla f=\lambda\nabla g$, and so

$$\left(2x-\frac{2-x-y}{2},2y-\frac{2-x-y}{2}\right)=\lambda\left(4x+1,4y+1\right), \qquad 2x^2+2y^2+x+y-2=0.$$

As usual, we have three equations in three variables. If you go through the painful process of solving them, you get: $x=-1,y=-1,\lambda=4/3$ or $x=1/2,y=1/2,\lambda=1/6$. These must be the max and min of the distance function, and so the points we're after are $(-1,-1,2)$ and $(1/2,1/2,1/2)$. The former maximizes distance $\sqrt{6}$, and the latter minimizes it $\sqrt{3}/2$.

There were a lot of slightly wrong ways to set up this problem, e.g. by only really using one constraint, or by maximizing $x^2+y^2+z^2$ instead of x^2+y^2. Unfortunately, a lot of these not-so-right methods would give the correct answer in the end. So don't be too taken aback if you got the right max but still lost some points.

(b) Compute the tangent lines to the ellipse at those points.

Solution:

The easiest way to get the tangent line is this case is to realize that it's the intersection of the tangent planes of the two surfaces at the points in question. The tangent plane to a plane at a point is easy: it's just the plane itself $x+y+2z=2$. For the tangent plane to the paraboloid, use the fact that it's orthogonal to the gradient of $z-x^2-y^2=0$, which is $(-2x,-2y,1)$. At $(-1,-1,2)$, this is $(2,2,1)$, and the plane is $2(x+1)+2(y+1)+(z-2)=0$, or $2x+2y+z=-2$. At $(1/2,1/2,1/2)$, the gradient is $(-1,-1,1)$, and the plane is $-(x-1/2)-(y-1/2)+(z-1/2)=0$, so $z-x-y=-1/2$.

This gives the implicit equation for the plane. At $(-1,-1,2)$, it's defined by $x+y+2z=2$ and $2x+2y+z=-2$. To get the explicit equation, take the cross product of the normal vectors: $(1,1,2)\times(2,2,1)=(-3,3,0)$. This means the line is $\vec{r}(t)=(-1-3t,-1+3t,2)$. Similarly, at $(1/2,1/2,1/2)$ the equations are $x+y+2z=2$ and $z-x-y=-1/2$, and cross product gives $\vec{r}(t)=(1/2+3t,1/2-3t,1/2)$.

A quicker way to find the tangent line is this. Think about the ellipse in the plane $2x^2+x+2y^2+y=2$ (this is the ellipse you get if you forget the z-coordinate). This is symmetric about the line $x=y$, and so the tangent directions at the two extreme points must both be $(1,-1)$. The tangent line to the ellipse at e.g. $(-1,-1,2)$ must be $(-1+t,-1-t,\text{something})$. But since distance is maximized, the tangent line has to be normal to the radial ector from $(-1,-1,2)$. This means z is a constant 2.

Problem 7 (*10+10 points*)**.** (a) Find the critical point of the function $f(x,y) = x^2 + xy + y^2 - 4x - 5y + 5$ and classify it.

Solution: From the system $\begin{cases} f_x(x,y) = 2x + y - 4 = 0 \\ f_y(x,y) = 2y + x - 5 = 0 \end{cases}$ we find that the critical point is $(1,2)$ is a critical point. For the Hessian we have: $f_{xx} = 2$, $f_{xy} = 1$ and $f_{yy} = 2$, hence $H(x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The determinant of this matrix is 3, and f_{xx} is grater than zero, hence the point $(1,2)$ is a minimum.

(b) For $x = t - s$, $y = t + s$, compute $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial s}$ and check that $\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} = 2\frac{\partial f}{\partial y}(t-s, t+s)$.

Solution: If we substitute $x(t,s)$ and $y(t,s)$ in $f(x,y)$ we obtain

$$f(x(t,s), y(t,s)) = 3t^2 + s^2 - 9t - s + 5.$$

Hence $\frac{\partial f}{\partial t} = 6t - 9$ and $\frac{\partial f}{\partial s} = 2s - 1$. Moreover $\frac{\partial f}{\partial y}(t, s, t + s) = 3t + s - 5$ and the claim follows.

Problem 8 (*10+5+5+5 points*)**.** This problem deals with the double integral

$$\int_{y=0}^{2}\int_{x=y}^{2} xy\,dx\,dy.$$

(a) Draw the region R over which the integral is being taken, and evaluate the double integral directly.

First we need to draw the region. It's illustrated below.

nn For the computation, we have

$$\int_{y=0}^{2}\int_{x=y}^{2} xy\,dx\,dy = \int_{y=0}^{2}\left(\left.\frac{x^2y}{2}\right|_{x=y}^{2}\right)dy = \int_{y=0}^{2}\left(2y-\frac{y^3}{2}\right)dy$$
$$= \left(\left.y^2-\frac{y^4}{8}\right|_{y=0}^{2}\right) = 4-2=2.$$

(b) Evaluate the integral by changing the order of integration, so that the inside integral is taken with respect to y.

Writing the bounds for the new order of integration,

$$I = \int_{x=0}^{2}\int_{y=0}^{x} xy\,dy\,dx = \int_{x=0}^{2}\left(\left.\frac{xy^2}{2}\right|_{y=0}^{x}\right)dx$$
$$= \int_{x=0}^{2}\frac{x^3}{2} = \left.\frac{x^4}{8}\right|_{x=0}^{2} = 2.$$

(c) Evaluate the integral by switching to polar coordinates.

Clearly we want θ to be the outside integral and range from 0 to $\pi/4$. For a given θ, we need r to go from 0 to $2\sec\theta$. The function is now

$$xy = r^2\cos\theta\sin\theta.$$

Plugging all that in and remembering that $dx\,dy = r\,dr\,d\theta$,

$$I = \int_{\theta=0}^{\pi/4}\int_{r=0}^{2\sec\theta} r^2\cos\theta\sin\theta\, r\,dr\,d\theta = \int_{\theta=0}^{\pi/4}\int_{r=0}^{2\sec\theta} r^3\cos\theta\sin\theta\,dr\,d\theta$$
$$= \int_{\theta=0}^{\pi/4}\frac{(2\sec\theta)^4}{4}\sin\theta\cos\theta\,d\theta = 2\int_{\theta=0}^{\pi/4}\frac{\sin\theta\cos\theta}{\cos^4\theta}\,d\theta$$
$$= 4\int_{\theta=0}^{\pi/4}\tan\theta\sec^2\theta d\theta = 2\left(\left.\frac{\sec^2\theta}{2}\right|_{0}^{\pi/4}\right) = 4(1-1/2)=2.$$

(d) Evaluate the integral using the change of coordinates $u = y/x$, $v = x$.

The three curves bounding our region are $u = 0$, $u = 1$, $v = 2$, and we see that the lower bound on v is 0. The function xy is uv^2.

Notice that $y = uv$, so the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ v & u \end{pmatrix} = -v.$$

The absolute value is v. So $dx\,dy = v\,du\,dv$.

So

$$I = \int_{v=0}^{2} \int_{u=0}^{1} uv^2\, v\,du\,dv = \int_{v=0}^{2} \left(\frac{u^2v^3}{2} \bigg|_{u=0}^{1} \right) dv = \int_{v=0}^{2} \frac{v^3}{2}\,dv = \frac{v^4}{8} \bigg|_{v=0}^{2} = 2.$$

Problem 9 *(5+5+10+5 points)*. Give bounds on integration for the following two- and three-dimensional regions, in the coordinate systems indicated.

(a) (Rectangular) The parallelogram with vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(2,1)$.

$$\int_{y=0}^{1} \int_{x=y}^{y+1} f(x,y)\,dx\,dy$$

or

$$\int_{x=0}^{1} \int_{y=0}^{x} f(x,y)\,dy\,dx + \int_{x=1}^{2} \int_{y=x-1}^{1} f(x,y)\,dy\,dx.$$

(b) (Polar) A circle centered at the point $(1,0)$, with radius 1.

$$\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} f(r,\theta)\,r\,dr\,d\theta$$

(Rectangular) The region bounded by the five planes $y=0$, $y=1$, $z=0$, $x=1$, and $x=z$.

$$\int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{x} f(x,y,z)\,dz\,dy\,dx.$$

(c) (Cylindrical, spherical) A cone with vertex at $(0,0,0)$ whose base is a circle of radius 2 centered at $(0,0,2)$ (parallel to the xy-plane).

Cylindrical:

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{h=r}^{2} f(r,\theta,h)\,r\,dh\,dr\,d\theta$$

Spherical:

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{2\sec\phi}$$

(d) Set up integrals to compute the z-coordinate of the center of mass of the cone from (c), assuming a uniform density $\delta(x,y,z)=1$. You don't have to actually compute it.

$$\text{mass} = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{h=r}^{2} 1\,r\,dh\,dr\,d\theta$$

$$z_{cm} = \frac{1}{\text{mass}} \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{h=r}^{2} h\,r\,dh\,dr\,d\theta$$