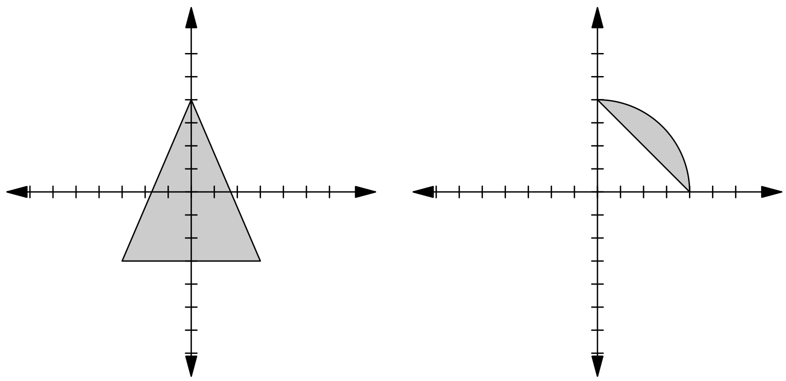
18.089 REVIEW OF MATHEMATICS

HOMEWORK 3, SOLS PART 2

Monday, July 1.

Exercise 1 ($\star$). *Write down bounds for integration over the following regions. Try to give both possible orders of integration.*



For the left one, suppose we want the integral dx on the outside. We'll need to break it into two regions, depending on which of the two sloped lines is the upper bound on y. From the slope and intercept, we can see that the two lines are $y = \frac{7}{3}x + 4$ (left), and $y = -\frac{7}{3}x + 4$ (right). So the integral is

$$\int_{x=-3}^{0} \int_{y=-3}^{\frac{7}{3}x+4} f(x,y)\,dy\,dx + \int_{x=0}^{3} \int_{y=-3}^{-\frac{7}{3}x+4} f(x,y)\,dy\,dx$$

Now let's do y as the outer variable. The bounds will again come from the equations for the two lines. Solving for x in the equations from earlier, the two lines are $x = (3y-12)/7$ and $x = -(3y-12)/7$. y itself ranges from -3 to 4, and so our integral is

$$\int_{y=-3}^{4} \int_{x=-(3y-12)/7}^{(3y-12)/7} f(x,y)\,dx\,dy.$$

For the second region, the lower bound on y is the line $y = 1-x$, and the upper bound is the circle, $y = \sqrt{1-x^2}$. So our integral is

$$\int_{x=0}^{1} \int_{y=1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

With the variables in the other order, it's similar: y will range from 0 to 1, and for a given y, the lower bound on x is $1-y$ and the upper bound is $\sqrt{1-y^2}$.

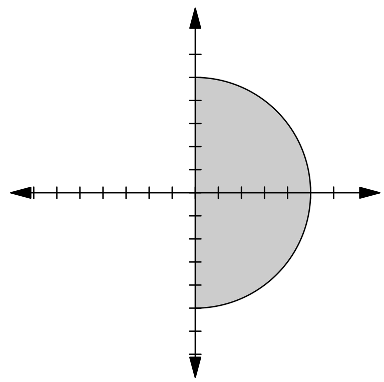
$$\int_{y=0}^{1} \int_{x=1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

Exercise 2. *Compute $\iint_R f(x,y)\,dA$, where R is the rectangle $-2 \le x \le 2$, $-1 \le y \le 1$ and $f(x,y) = xe^{xy}$. One of the possible orders of integration will be easier – which one?*

Notice that the given function is just $\frac{\partial}{\partial y} e^{xy}$, so integrating dy is going to be easy. We set it up with that as the inside integral. (Integrating dx isn't so hard either: just need to use integration by parts once).

$$\int_{x=-2}^{2}\int_{y=-1}^{1} xe^{xy}\,dy\,dx = \int_{x=-2}^{2} (e^{xy})|_{-1}^{1} = \int_{x=-2}^{2} (e^x - e^{-x})\,dx$$
$$= (e^x + e^{-x})|_{-2}^{2} = e^2 + e^{-2} - e^{-2} - e^2 = 0.$$

Exercise 3 ($\star$)**.** *Find the center of mass of the following semicircular region, assuming the density is a uniform $\delta(x, y) = 1$. The center of the circle is $(0, 0)$, and the radius is 5.*



Yeah, this is easier with polar coordinates, but let's use Cartesian since that's what was covered this day. It's clear from symmetry that the y-coordinate of the center of mass is 0. First we need to compute

$$\iint_R x\delta(x, y)\,dA = \int_{x=0}^{5}\int_{y=-\sqrt{25-x^2}}^{\sqrt{25-x^2}} x\,dy\,dx = \int_{x=0}^{5} 2x\sqrt{25-x^2}\,dx.$$

Take $u = 25 - x^2$, so $du = -2x\,dx$. The bound $x = 0$ becomes $u = 25$, and $x = 5$ is $u = 0$.

$$= \int_{u=25}^{0} (-\sqrt{u}\,du) = \int_{u=0}^{25} \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^{25} = \frac{250}{3}.$$

The total mass is density times area, which is $\frac{25\pi}{2}$. So we have

$$x_{cm} = \frac{250/3}{2/25\pi} = \frac{20}{3\pi} \approx 2.12,$$

which is a bit to the left of the half-way mark $x = 2.5$, and seems plausible since the thing has more mass to the left.

Exercise 4 ($\star$)**.** *Compute*

$$\int_{x=-\infty}^{\infty}\int_{y=1}^{2} \frac{1}{(x^2+y^2)^2}\,dy\,dx$$

by switching to polar coordinates. Hint: the trig identity $\sin^2 t = \frac{1-\cos 2t}{2}$ *just might come in handy.*

To find the bound on θ for a given r, draw the region, a ray at angle θ, and look for a right triangle (let me know if you don't see what I'm doing here, I'm happy to scan a picture).

$$I = \int_{\theta=0}^{2\pi}\int_{r=\csc\theta}^{2\csc\theta} \frac{1}{r^4} r\,dr\,d\theta = \int_{\theta=0}^{\pi} \left(-\frac{1}{2r^2}\Big|_{r=\csc\theta}^{2\csc\theta} \right)$$
$$= \int_{\theta=0}^{\pi} \frac{3}{8}\sin^2 t\,dt = \frac{3\pi}{16}.$$

Exercise 5. *Find the center of mass of the northern hemisphere, assuming that the earth is a sphere of radius 4,000 miles and uniform density (not a very good assumption), centered at $(0, 0, 0)$. What line of latitude has the same z-coordinate as this point?*

Let $R = 4000$. We set this up as a triple integral in spherical coordinates. The total mass is $\frac{1}{2}\frac{4}{3}\pi R^3 = \frac{2}{3}\pi R^3$. The function z is $\rho\cos\phi$

$$\begin{aligned} m \cdot z_{cm} &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{R} (\rho\cos\phi)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta \\ &= \frac{R^4}{4} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \cos\phi \sin\phi\, d\phi\, d\theta \\ &= \frac{R^4}{8} \int_{\theta=0}^{2\pi} d\theta = \frac{\pi R^4}{4}. \end{aligned}$$

Dividing by mass,

$$z_{cm} = \frac{3R}{8}.$$

If $R = 4000$, this is 1500 mi. The line of latitude is $\sin^{-1}(3/8) \approx 22°$ N.

Exercise 6 ($\star$). *Find the volume of the region in space bounded on the sides by the xz- and yz-planes, below by the plane $x + y + z = 1$, and above by the plane $x + y + z/2 = 1$. It will be helpful to sketch this region first!*

The region between these planes is a tetrahedron which lies over a triangle in the plane bounded by the two axes and the line $x + y = 1$. The lower bound on z comes from the first plane, and the upper bound on z comes from the second plane.

$$\begin{aligned} V &= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=1-x-y}^{2-2x-2y} 1\, dz\, dy\, dx = \int_{x=0}^{1} \int_{y=0}^{1-x} 1 - x - y\, dy\, dx \\ &= \int_{x=0}^{1} (1-x)(1-x) - \frac{(1-x)^2}{2}\, dx = \frac{1}{6}. \end{aligned}$$

Wednesday, July 3.

Exercise 7 ($\star$). *Let R be a parallelgram with vertices at $(0,0)$, $(1,2)$, $(2,1)$, and $(3,3)$. Compute*

$$\iint_R 3x\, dx\, dy$$

using an appropriate change of coordinates.

The lines bounding this parallelogram have the following equations: left, $2x - y = 0$; right, $2x - y = 3$; bottom, $2y - x = 0$; top, $2y - x = 3$. So let's use $u = 2x - y$, $v = 2y - x$.

Our bounds will now be $u = 0$, $u = 3$, $v = 0$, $v = 3$. Our function is x can be written as $2u + v$. Note that

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3,$$

and so

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \frac{1}{3}$$

which means that

$$dx\, dy = \frac{1}{3}\, du\, dv.$$

At last,

$$\iint_R 3x\, dx\, dy = \int_{u=0}^{3} \int_{v=0}^{3} (2u + v)\frac{1}{3}\, dv\, du = \cdots = \frac{27}{2}.$$

Exercise 8 ($\star$). *Give bounds for triple integrals over the following regions, using the specified coordinates:*

- *Cylindrical: a cone whose base is the unit circle in the xy-plane, with the point of the cone at $(0, 0, 3)$.*
$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{h=0}^{3-3r} f(r,\theta,h)\,r\,dh\,dr\,d\theta.$$
- *Spherical: the first octant of a sphere of radius 1 centered at the origin (i.e. the eighth of the sphere where x, y, and z are all positive).*
$$\int_{\theta=0}^{\pi/4}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{1} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$
- *Both: the region between the planes $z = 1$ and $z = 2$.*
  Cylindrical first:
$$\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}\int_{h=1}^{2} f(r,\theta,h)\,r\,dh\,dr\,d\theta.$$
  Now spherical:
$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=\sec\theta}^{2\sec\theta} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$