18.089 Exam 1

Handed out Friday, June 14, 2013; due in class Monday, June 17, 2013

Name: Solutions

This exam consists of eight problems, not arranged in any particular order. Please solve all problems in the space provided (or attaching additional sheets as necessary), showing all work as neatly and clearly as possible.

All work must be your own. In particular, you may not seek help from other students in the class or any "live" internet resources. But feel free to use reference works such as your class notes, a textbook, or Wikipedia. If you have any questions about what constitutes an acceptable resource, please ask.

Problem	**Value**	**Score**
Problem 1	5	
Problem 2	10	
Problem 3	15	
Problem 4	10	
Problem 5	10	
Problem 6	15	
Problem 7	15	
Problem 8	20	
Total	100	

Problem 1. *(5 points)* Compute the derivative $\frac{d}{dx}\ln(3^x+1)$.

Solution. By the chain rule, we have

$$\frac{d}{dx}\ln(u) = \frac{1}{u}\cdot\frac{du}{dx},$$

where in this case $u = 3^x + 1$. Now using our rule for derivatives of exponential functions (or rederiving the rule using the algebraic manipulation $3^x = e^{x\ln 3}$) we have that $\frac{du}{dx} = 3^x\ln(3)$. Putting this all together, we conclude that

$$\boxed{\frac{d}{dx}\ln(3^x+1) = \frac{3^x\ln(3)}{3^x+1}.}$$

Problem 2. *(10 points)* What is the third derivative of $f(x) = \sin x\cos x$?

Solution. Method 1: By the product rule,

$$f'(x) = \sin x\cdot(-\sin x) + \cos x\cdot\cos x = \cos^2 x - \sin^2 x,$$

so by the chain rule

$$f''(x) = 2\cos x\cdot(-\sin x) - 2\sin x\cos x = -4\sin x\cos x.$$

Thus

$$\boxed{f'''(x) = -4(\cos^2 x - \sin^2 x) = 4\sin^2 x - 4\cos^2 x.}$$

Method 2: Observe that $2f(x) = 2\sin x\cos x = \sin(2x)$, so $f(x) = \frac{1}{2}\sin(2x)$. Thus

$$\begin{aligned} f'(x) &= \frac{1}{2}\cdot 2\cos(2x) = \cos(2x), \\ f''(x) &= -2\sin(2x) \end{aligned}$$

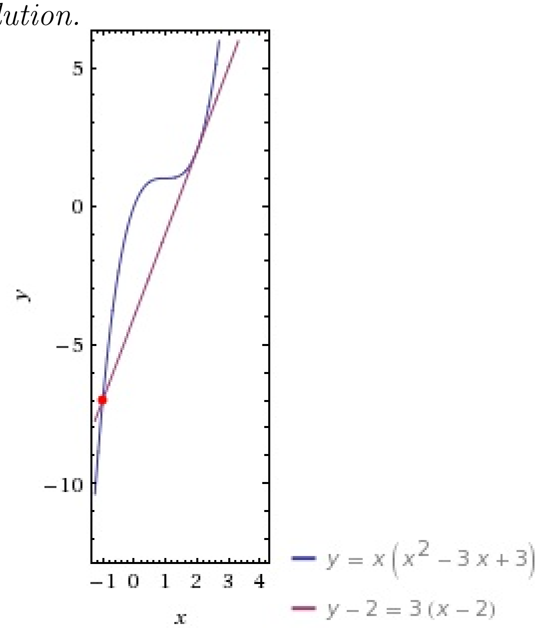
and

$$\boxed{f'''(x) = -4\cos(2x).}$$

(Several other equivalent forms of the final answer are also possible, for example $4 - 8\cos^2(x)$.)

Problem 3. *(15 points)* What is the tangent line to the curve $y = x^3 - 3x^2 + 3x$ at $x = 2$? Compute the area of the region between the curve and the tangent line.

Solution.



Let's first find the slope of the tangent line.

$$y'(x) = 3x^2 - 6x + 3 \quad \text{and} \quad y'(2) = 3 \times 2^2 - 6 \times 3 + 3 = 12 - 18 + 3 = 3.$$

Thus the equation of the line is

$$y - y(2) = 3(x-2) \Leftrightarrow y - (8 - 12 + 6) = 3x - 6 \Leftrightarrow y = 3x - 4.$$

Next, we need to find the intersection points between the tangent line and the curve.

$$3x - 4 = x^3 - 3x^2 + 3x \Leftrightarrow x^3 - 3x^2 + 4 = 0 \Leftrightarrow (x-2)^2(x+1) = 0$$

where we made use of the fact that we already knew that 2 would be a double root of the equation above (because of tangency). Thus the area we want is

$$\int_{-1}^{2} x^3 - 3x^2 + 3x - (3x - 4)\, dx = \int_{-1}^{2} x^3 - 3x^2 + 4\, dx = \frac{x^4}{4} - x^3 + 4x\Big]_{-1}^{2}$$
$$= \frac{16}{4} - 8 + 8 - \frac{1}{4} - 1 + 4 = \frac{27}{4}.$$

Problem 4. *(10 points)* Compute $\displaystyle\int_0^\pi x^2 \sin x\,dx$. (Cryptic hint: if once isn't enough, try it again.)

Solution. To save space, write $I = \int_0^\pi x^2 \sin x\,dx$. We integrate by parts. Set $u = x^2$, $dv = \sin x\,dx$, so $du = 2x\,dx$ and $v = -\cos x$. Then

$$\begin{aligned}
I &= -x^2 \cos x\Big]_0^\pi - \int_0^\pi -2x\cos x\,dx \\
&= (-\pi^2 \cdot (-1) - 0) + 2\int_0^\pi x\cos x\,dx \\
&= \pi^2 + 2\int_0^\pi x\cos x\,dx.
\end{aligned}$$

Following the cryptic hint, we integrate by parts again, taking $u = x$, $dv = \cos x\,dx$, so $du = dx$ and $v = \sin x$. This gives

$$\begin{aligned}
I &= \pi^2 + 2\left(x\sin x\Big]_0^\pi - \int_0^\pi \sin x\,dx\right) \\
&= \pi^2 + 2\left((\pi\cdot 0 - 0\cdot 0) - \left(-\cos x\Big]_0^\pi\right)\right) \\
&= \pi^2 + 2(-1-1) \\
&= \pi^2 - 4.
\end{aligned}$$

Problem 5. *(10 points)* Does $\displaystyle\int_1^\infty \frac{1}{x^4+1}$ converge or diverge? Explain.

Solution. The dominant term in the denominator is x^4, and this leads us to conjecture that the integral does converge. To prove this, we use a comparison test: since $x^4 < x^4 + 1$ for all x, we have

$$\frac{1}{x^4} > \frac{1}{x^4+1}$$

and thus if

$$\int_1^\infty \frac{1}{x^4}\,dx$$

converges then our integral must as well. The convergence of this latter integral was discussed in class; since $4 > 1$, it converges.

Some students also successfully did a comparison test with the integrals $\int_1^\infty \frac{1}{x^3}\,dx$ and $\int_1^\infty \frac{1}{x^2+1}\,dx$.

Problem 6. *(15 points)* Compute $\displaystyle\int \frac{1}{x^3 + 4x^2 + 5x}\, dx$.

Solution. Let's start by factorizing the denominator, and completing square where appropriate:

$$x^3 + 4x^2 + 5x = x(x^2 + 4x + 5) = x((x + 2)^2 + 1).$$

Thus we have the partial fraction form:

$$\frac{1}{x^3 + 4x^2 + 5x} = \frac{A}{x} + \frac{B(x + 2) + C}{((x + 2)^2 + 1}$$

where A, B and C are constants to be found. Using the cover-up method for A gives

$$\frac{1}{x^2 + 4x + 5} = \frac{A}{1} + \frac{x(B(x + 2) + C)}{((x + 2)^2 + 1}$$

and setting $x = 0$, we get $A = \frac{1}{5}$. To get B and C, one could use the "complex cover-up method" (see addendum to lecture 3), or alternatively, just use substraction:

$$\begin{aligned}\frac{B(x + 1) + C}{((x + 2)^2 + 1} &= \frac{1}{x^3 + 4x^2 + 5x} - \frac{1}{5x} = \frac{5 - ((x + 2)^2 + 1)}{5(x^3 + 4x^2 + 5x)} = \frac{5 - ((x + 2)^2 + 1)}{5(x^3 + 4x^2 + 5x)} \\ &= \frac{-x^2 - 4x}{5x((x + 2)^2 + 1)} = \frac{-(x + 2)/5 - 2/5}{(x + 2)^2 + 1}\end{aligned}$$

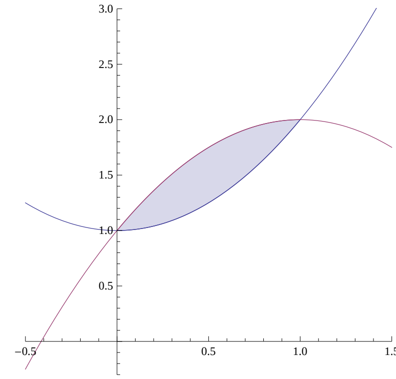
Altogether, we have

$$\frac{1}{x^3 + 4x^2 + 5x} = \frac{1/5}{x} + \frac{-(x + 2)/5}{(x + 2)^2 + 1} + \frac{-2/5}{(x + 2)^2 + 1}$$

and integrating gives

$$\int \frac{1}{x^3 + 4x^2 + 5x}\, dx = \frac{1}{5}\ln|x| - \frac{1}{10}\ln((x + 2)^2 + 1) - \frac{2}{5}\arctan(x + 2) + C.$$

Problem 7. *(15 points)* The region above the curve $y = x^2 + 1$ and below the curve $y = -x^2 + 2x + 1$ is rotated around the x-axis. (Just for emphasis: we are rotating around the x-axis, NOT the y-axis.) Compute the volume of the resulting solid.



Solution. The diagram is at right. We begin by finding the boundary of the region in question. The x-coordinates of the intersection points of the two curves are the solutions of the equation $x^2 + 1 = -x^2 + 2x + 1$. Collecting terms, this simplifies to $x(x-1) = 0$, with solutions $x = 0$ and $x = 1$, and corresponding y-values 1 and 2. Moreover, we can see from the diagram that the upward-facing parabola forms the lower and right boundaries of the region, while the downward-facing parabola form sthe upper and left boundaries.

Method 1: by disks. To use the disk method, we realize that we have to subtract one solid from the other. The larger solid is what we get when we rotate the region bounded above by $y = -x^2 + 2x + 1$ between $x = 0$ and $x = 1$ around the x-axis. The area of a cross-section of this solid is $\pi(-x^2 + 2x + 1)^2$, and so the volume is

$$\begin{aligned}\int_0^1 \pi(-x^2 + 2x + 1)^2\,dx &= \pi \int_0^1 x^4 - 4x^3 + 2x^2 + 4x + 1\,dx \\ &= \pi \left(\frac{x^5}{5} - x^4 + \frac{2x^3}{3} + 2x^2 + x \Big]_0^1 \right) \\ &= \frac{43\pi}{15}.\end{aligned}$$

x

The smaller solid we must remoive is what we get when we rotate the region bounded above by $y = x^2 + 1$ between $x = 0$ and $x = 1$ around the x-axis. The area of a cross-section of this solid is $\pi(x^2 + 1)^2$, and so the volume is

$$\begin{aligned}\int_0^1 \pi(x^2 + 1)^2\,dx &= \pi \int_0^1 x^4 + 2x^2 + 1\,dx \\ &= \pi \left(\frac{x^5}{5} + \frac{2x^3}{3} + x \Big]_0^1 \right) \\ &= \frac{28\pi}{15}.\end{aligned}$$

The volume of the requested solid is the difference $\frac{43\pi}{15} - \frac{28\pi}{15} = \pi$.

Method 2: with shells. To use shells, we must integrate with respect to y. This means we have to express the boundaries with x in terms of y. In our case, we can use the quadratic formula to find that we have right boundary $x = \sqrt{y-1}$ and left boundary $x = 1 - \sqrt{2-y}$. (We figure out which root to choose by comparing with the diagram.) Then a single shell has radius y and height $\sqrt{y-1} - (1 - \sqrt{2-y})$ and so has volume $2\pi y(\sqrt{y-1} + \sqrt{2-y} - 1)\,dy$. It follows that the total volume is

$$\int_1^2 2\pi y(\sqrt{y-1} + \sqrt{2-y} - 1)\,dy.$$

The easiest way to compute this integral is probably to split it up into three pieces and use the substitution $u = y - 1$ in one piece and $u = 2 - y$ in the second.

Problem 8. *(20 points)* Compute $\int \cos^4 x\,dx$ and $\int \cos^5 x\,dx$.

Solution. There are multiple ways of approaching this (and they give seemingly different answers, though you can check that they are actually the same function, expressed differently). Many people made use of the identity:

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

which can be found in various references. Here are some alternative solutions.

$$\begin{aligned}
\cos^4(x) &= \cos^2(x) \times \cos^2(x) \\
&= \frac{1+\cos(2x)}{2} \times \frac{1+\cos(2x)}{2} \\
&= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x) \\
&= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\Big(\frac{1+\cos(4x)}{2}\Big) \\
&= \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)
\end{aligned}$$

Integrating, we get:

$$\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$$

$$\begin{aligned}
\cos^5(x) &= \cos^4(x) \times \cos(x) \\
&= \frac{3}{8}\cos x + \frac{1}{2}\cos(2x)\cos x + \frac{1}{8}\cos(4x)\cos x \\
&= \frac{3}{8}\cos x + \frac{1}{2} \times \frac{1}{2}(\cos(x) + \cos(3x)) + \frac{1}{8} \times \frac{1}{2}(\cos(3x) + \cos(5x)) \\
&= \frac{5}{8}\cos x + \frac{5}{16}\cos(3x) + \frac{1}{16}\cos(5x)
\end{aligned}$$

Integrating, we get:

$$\int \cos^5 x\,dx = \frac{5}{8}\cos x + \frac{5}{54}\sin(2x) + \frac{1}{80}\sin(4x) + C.$$