18.086 Project Report

Perfect Matched Layer for Acoustic Wave Equations

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Introduction

Perfect Matched Layer (PML) is an important kind of numerical boundary condition in wave propagation simulations, which is always applied to mimic the infinite large space or in situations of perfect absorption in reality. The first study about PML dates back to 1994 [1]. In this work, the author constructed a PML for time dependent Maxwell equations using field split method. After that, different PMLs (split-field PML, uniaxial PML, and coordinate transformation PML) has been constructed for both electromagnetic wave [2][3][4]and acoustic wave problems[2][5][6].

In this course project, I study the PML in 2-D acoustic wave propagation, and explore and evaluate the factors that affect the PML performance.

System model and acoustic wave equations

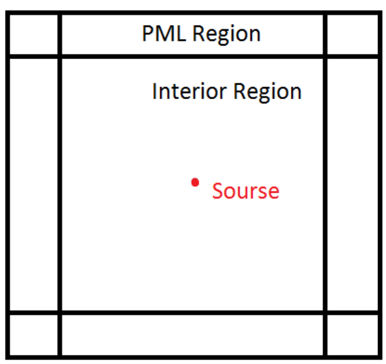

Fig.1 System model

The main objective of this project is to numerically construct perfect matched layers around a 2-D square region to reduce the reflection of the acoustic wave from a point source in the middle of this region (Fig. 1). The width of interior region is 1m, and the width of PMLs is 0.2m. The interior region is filled with air (sound velocity $c = 340m/s$, density $\rho = 1.2\text{kg}/m^3$).

The acoustic wave equations in air are as follows:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\partial p}{\partial t} = -\rho c^2 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}),$$

(1)

where p is the perturbed pressure and u, v are perturbed velocities in x and y directions respectively.

The point source locates at the center of the square region. The pressure of this source is a sinusoidal function of time.

PML construction

The PML layers are constructed by adding dissipative terms in order to attenuate outgoing waves exponentially, and in the meanwhile, the wave should be continuous at the interface of the interior region and the PML region. For the 2-D problem here, two attenuation coefficients, σ_x and σ_y , are introduced into the equation according to [1][2]. σ_x and σ_y increase gradually from the inner boundary to the outer boundary of the PML layers. That is to say, $\sigma_x = \frac{x_i}{w_{PML}} \sigma_{max}$, $\sigma_y = \frac{y_i}{w_{PML}} \sigma_{max}$, where x_i and y_i are distances to the inner boundary of PML layers, w_{PML} is the width of the PML layer, which is 0.2m here, and σ_{max} is the largest value of the attenuation coefficient.

After the transformation, the acoustic wave equations become [2]

$$\frac{\partial u}{\partial t} + c\sigma_x u = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$
$$\frac{\partial v}{\partial t} + c\sigma_y u = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$
$$\frac{\partial p}{\partial t} + c(\sigma_x + \sigma_y)p = -\rho c^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \rho c^2 \sigma_x \frac{\partial Q}{\partial y} - \rho c^2 \sigma_y \frac{\partial R}{\partial y},$$
$$\frac{\partial Q}{\partial t} = cv\ ,$$
$$\frac{\partial R}{\partial t} = cu\ .$$

(2)

In the two PML layers at x –axis side, $\sigma_x > 0\ and\ \sigma_y = 0$. In the other two PML layers at y-axis side, $\sigma_y > 0\ and\ \sigma_x = 0$.

Computation

The equations (2) are a set of partial differential equations. A central difference scheme is applied to discretize the 2-D first order spacial derivative and the Leapfrog method is applied to the first order time derivative. The total accuracy of the computational method is of second order.

For the central difference in space, we have

$$\frac{\partial p}{\partial x} = \frac{\mathrm{p}_{\mathrm{j+1,k}} - p_{j-1,k}}{2\Delta x} = \nabla_{\mathrm{x}} p$$

$$\frac{\partial p}{\partial y} = \frac{\mathrm{p}_{\mathrm{j,k+1}} - p_{j,k-1}}{2\Delta y} = \nabla_{\mathrm{y}} p$$

Similar expressions can be obtained for the first order derivatives of u, v, Q, and R in (2).

For the Leapfrog scheme in time domain, we have

$$\frac{\mathrm{u}^{\mathrm{n+1}} - \mathrm{u}^{\mathrm{n}}}{\Delta t} + c\sigma_x \mathrm{u}^{\mathrm{n}} = -\frac{1}{\rho} \nabla_{\mathrm{x}} p^{n+\frac{1}{2}},$$

$$\frac{\mathrm{v}^{\mathrm{n+1}} - \mathrm{v}^{\mathrm{n}}}{\Delta t} + c\sigma_y \mathrm{v}^{\mathrm{n}} = -\frac{1}{\rho} \nabla_{\mathrm{y}} p^{n+\frac{1}{2}},$$

$$\frac{\mathrm{Q}^{\mathrm{n+1}} - \mathrm{Q}^{\mathrm{n}}}{\Delta t} = c v^{n} \ ,$$

$$\frac{\mathrm{R}^{\mathrm{n+1}} - \mathrm{R}^{\mathrm{n}}}{\Delta t} = c u^{n},$$

$$\frac{\mathrm{p}^{\mathrm{n}+\frac{3}{2}} - \mathrm{p}^{\mathrm{n}+\frac{1}{2}}}{\Delta t} + c(\sigma_x + \sigma_y)\mathrm{p}^{\mathrm{n}+\frac{1}{2}} = -\rho c^2 \left(\nabla_{\mathrm{x}} \mathrm{u}^{\mathrm{n+1}} + \nabla_{\mathrm{y}} \mathrm{v}^{\mathrm{n+1}}\right) - \rho c^2 \sigma_x \nabla_{\mathrm{y}} \mathrm{Q}^{\mathrm{n+1}} - \rho c^2 \sigma_y \nabla_{\mathrm{x}} \mathrm{R}^{\mathrm{n+1}}.$$

Results

In all the simulations below, the time dependence of perturbed pressure at the point source is

$p_0 = \cos(2\pi f t)$,

where $f = 2720Hz$ （wavelength $\lambda = \frac{c}{f} = \frac{340}{2720} = 0.125\mathrm{m}$）.

In the discretization scheme, Δx =Δy=1/200 m, Δt=1/13600000 s.

First, I simulate the pressure profile p at different times $\mathrm{t} = \frac{2}{\mathrm{f}}, \frac{4}{\mathrm{f}}, and\, \frac{6}{f}$, when the largest attenuation coefficient $\sigma_{max} = 0, 5, 50, and\ 100$. The results are shown in Fig. 2.

We can see that when $\mathrm{t} \leq \frac{2}{\mathrm{f}}$, the wavefront haven't propagate to the interface between PMLs and the interior region. When $\mathrm{t} > \frac{2}{\mathrm{f}}$, the wavefront reaches the interface. When $\sigma_{max} = 0\ and\ 5$, there is no attenuation or small attenuation in the PMLs, thus we still see reflected waves backward the interior region. When $\sigma_{max} = 50\ and\ 100$, there is little reflection at the interface of PML and interior region. The outward wave is almost dissipated by the PMLs.

Two movies for $\sigma_{max} = 0\ and\ 100$ are attached to show the effectiveness of the PMLs in this simulation.

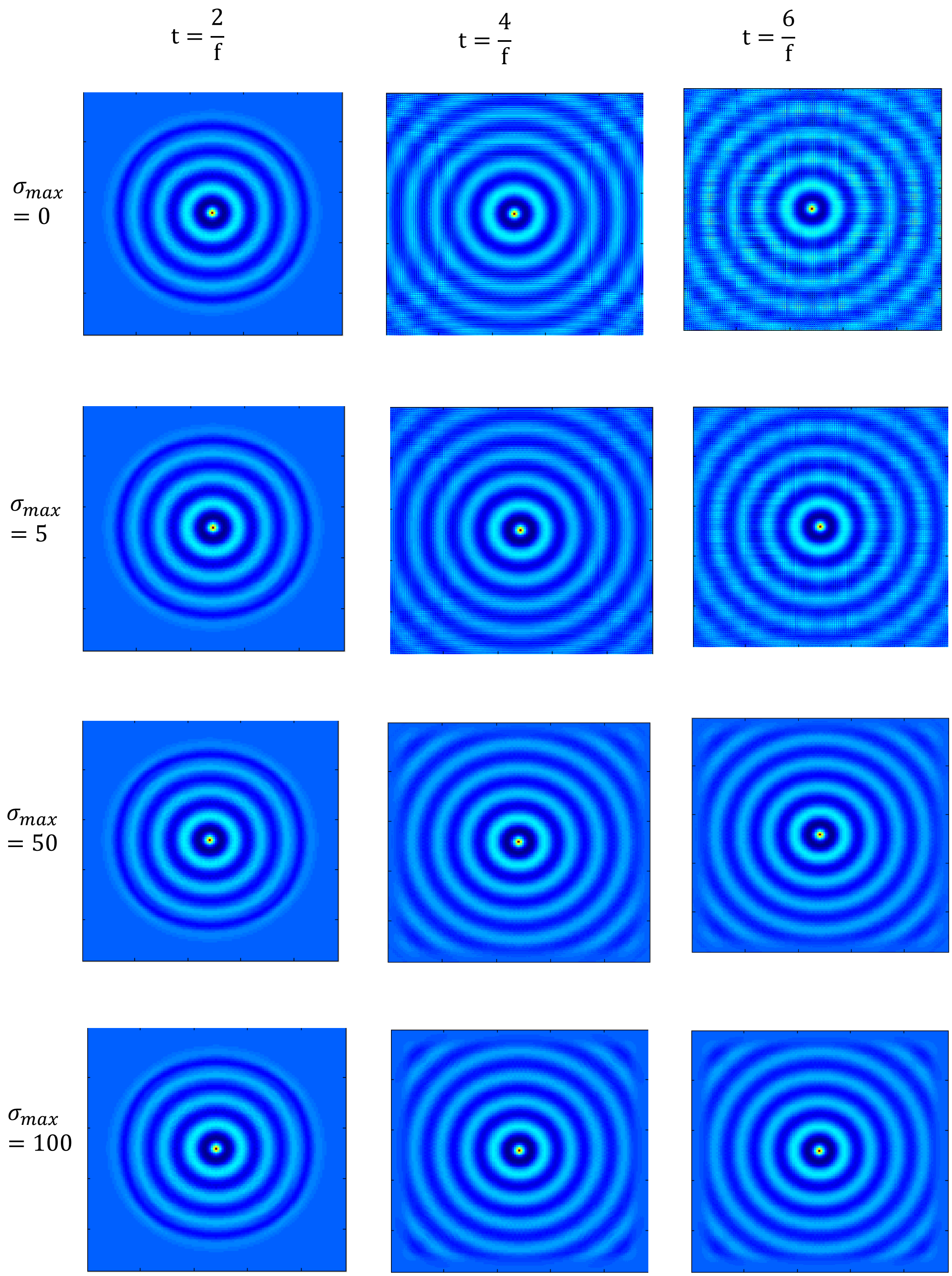

Fig. 2 Simulation results at different time t for different σ_{max}

Discussions

1. The PMLs are only reflectionless for exact continuous wave equation. But in any computer simulations, the wave equation is discretized, so there will be numerical reflections [7]. The reason the attenuation coefficient is set to increase from the inner boundary to the outer boundary is to reduce this numerical reflection.
2. The analysis shown in this project can only be applied to Cartesian coordinates. The wave equations and the attenuation coefficient must be adjusted in order to be applied to other coordinate systems.
3. There is a trade-off between the thickness of the PMLs and the σ_{max}. On one hand, for thinner PMLs, we need larger σ_{max} in order to achieve better attenuation. However, this also means the gradient of σ_{max} from inner boundary to outer boundary is larger, which will generate more numerical reflection. On the other hand, thicker PMLs need more simulation time and computer memory. We always need to find a balance between these two factors.
4. In the acoustic wave equations after transformation, the time derivative becomes

$$\frac{\partial u}{\partial t} + c\sigma_x u = (iw + c\sigma_x)u$$
$$\frac{\partial v}{\partial t} + c\sigma_y v = (iw + c\sigma_y)v$$

   If we consider the 1-D case, then

$$(iw + c\sigma_x)u = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$(iw + c\sigma_x)p = -\rho c^2 \frac{\partial u}{\partial x}$$

   Then we have

$$\frac{\partial^2 p}{\partial x^2} + k_c^2 p = 0$$

   Where $k_c = k - i\sigma_x$,

   In which σ_x is the term that causes dissipation. We can see that the dissipation is frequency independent.

5. In positive index materials, if the phase velocity has positive sign, and if $\sigma_x > 0$ and $\sigma_y > 0$, the outgoing wave is attenuated in PML. However, in negative index metamaterials, the outgoing wave has negative phase velocity, thus will be amplified by the PML. The PML lost its effect in this case.

Conclusion

In this project, I studied the Perfect Matched Layers (PML) and their application in acoustic wave simulations. PMLs for a square region are constructed and tested under a point source in the center of the region. Results show that the PML I constructed can effectively attenuate the outgoing wave and generate little reflection.

References

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