MIT Dept. of Mathematics
Benjamin Seibold

18.086 spring 2008
Exercise Sheet 5

Out Mon 04/18/08 Due Fri 05/02/08

Exercise 10 Consider the heat equation on a rod of length π, which has a fixed temperature at both ends.

$$\begin{cases} u_t = u_{xx} & \text{for } (x,t) \in\,]0,\pi[\times]0,t_f] \\ u(x,0) = u_0(x) & \text{for } x \in [0,\pi] \\ u(0,t) = u(\pi,t) = 0 & \text{for } t \in\,]0,t_f] \end{cases}$$

Let the final time be $t_f = 0.1$. We can measure the temperature distribution at the final time $u(x,t_f)$, and would like to reconstruct the initial temperature distribution $u(x,0)$ from it.

1. Explain why this problem is ill-posed. Give an example of a function $u(x,t_f)$ which no valid initial function $u(x,0)$ exists for.
2. We approximate the problem by a finite dimensional problem, by considering a finite number of Fourier coefficients

$$u(x,t) = \sum_{k=1}^{N} c_k(t)\sin(kx) \tag{1}$$

   At a given time t, we can then write the solution as a vector

$$\mathbf{c}(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_N(t) \end{pmatrix}$$

   Give an expression for the coefficients $c_k(t)$, using the initial coefficients $c_k(0)$. What is the relation between $||u(\cdot,t)||_{L^2([0,\pi])} = (\int_0^\pi u(x,t)^2 dx)^{\frac{1}{2}}$ and $||\mathbf{c}(t)||_2$?
3. Give the matrix A_t which maps $\mathbf{c}(0)$ to $\mathbf{c}(t)$, i.e. $\mathbf{c}(t) = A_t \cdot \mathbf{c}(0)$. Approximate your example function from part 1 by a finite number of Fourier coefficients, using A_{t_f} to compute $\mathbf{c}(0)$. Compute and plot the approximate functions $u(x,t_f)$ and $u(x,0)$ using formula (1).
4. Fix a number of Fourier coefficients N, and choose the initial condition to be $c_k(0) = \frac{1}{Z}\exp(-\frac{k}{4})$, where the constant Z is chosen, such that $||u(\cdot,t_f)||_{L^2([0,\pi])} =$

$\sqrt{\frac{\pi}{2}}$. We try to measure the exact vector $\mathbf{c}(t_f)$, but the measurement involves an error $\mathbf{e}$, so we actually measure $\mathbf{c}^e(t_f) = \mathbf{c}(t_f) + \mathbf{e}$. Choose the error to be of the form $\tilde{e}_k = \exp(-\frac{k}{4})\text{randn}$, and then scale the components $e_k = \frac{1}{Z_e}\tilde{e}_k$, such that $||\mathbf{e}||_2 = \delta||\mathbf{c}(t_f)||_2$.
Give a formula for the reconstruced initial vector $\mathbf{c}^e(0) = A_{t_f}^{-1} \cdot \mathbf{c}^e(t_f)$ and the thus made error $||\mathbf{c}^e(0) - \mathbf{c}(0)||_2$. Explain why the problem requires regularization.

5. Let $\mathbf{c}^{e,\alpha}(0)$ denote the solution to the Tychonov regularized backwards problem

$$\mathbf{c}^{e,\alpha}(0) = (A^T A + \alpha I)^{-1} A^T \cdot \mathbf{c}^e(t_f)$$

   Compute the error to the correct initial condition $||\mathbf{c}^{e,\alpha}(0) - \mathbf{c}(0)||_2$ in dependence on the regularization parameter α, and find the $\hat{\alpha}$ which this error becomes minimal for. You can to this either by hand (doable, but technical) or by writing a matlab program, which runs though different values of α and finds the minimizer. Plot the error as a function of α for interesting values of N and δ.

6. For $N = 5, 10, 15$, produce plots of the optimal $\hat{\alpha}$, which minimizes the error, in dependence on the relative error size δ. Again, this can be computed by hand, or in matlab, by running step 5 for a list of values for δ. Compare the results to the theoretical estimate for the optimal α, provided in Section 8.2 in Strang's CSE book. Explain possible deviations.