Exercise 10 Consider the heat equation on a rod of length $\pi$, which has a fixed temperature at both ends.

$$\begin{align*}
\left\{ \begin{array}{ll}
u_t = u_{xx} & \text{for } (x,t) \in [0,\pi] \times [0,t_f] \\
u(x,0) = u_0(x) & \text{for } x \in [0,\pi] \\
u(0,t) = u(\pi,t) = 0 & \text{for } t \in [0,t_f]
\end{array} \right.
\end{align*}$$

Let the final time be $t_f = 0.1$. We can measure the temperature distribution at the final time $u(x,t_f)$, and would like to reconstruct the initial temperature distribution $u(x,0)$ from it.

1. Explain why this problem is ill-posed. Give an example of a function $u(x,t_f)$ which no valid initial function $u(x,0)$ exists for.

2. We approximate the problem by a finite dimensional problem, by considering a finite number of Fourier coefficients

$$u(x,t) = \sum_{k=1}^{N} c_k(t) \sin(kx) \quad (1)$$

At a given time $t$, we can then write the solution as a vector

$$c(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_N(t) \end{pmatrix}$$

Give an expression for the coefficients $c_k(t)$, using the initial coefficients $c_k(0)$. What is the relation between $\|u(\cdot,t)\|_{L^2([0,\pi])} = (\int_0^\pi u(x,t)^2 dx)^{1/2}$ and $\|c(t)\|_2$?

3. Give the matrix $A_t$ which maps $c(0)$ to $c(t)$, i.e. $c(t) = A_t \cdot c(0)$. Approximate your example function from part 1 by a finite number of Fourier coefficients, using $A_{t_f}$ to compute $c(0)$. Compute and plot the approximate functions $u(x,t_f)$ and $u(x,0)$ using formula (1).

4. Fix a number of Fourier coefficients $N$, and choose the initial condition to be $c_k(0) = \frac{1}{Z} \exp(-\frac{k^2}{4})$, where the constant $Z$ is chosen, such that $\|u(\cdot,t_f)\|_{L^2([0,\pi])} =$
\[ \sqrt{2}. \] We try to measure the exact vector \( c(t_f) \), but the measurement involves an error \( e \), so we actually measure \( c^e(t_f) = c(t_f) + e \). Choose the error to be of the form \( \tilde{e}_k = \exp(-\frac{k}{4}) \text{randn} \), and then scale the components \( e_k = \frac{1}{Z_e} \tilde{e}_k \), such that \( ||e||_2 = \delta ||c(t_f)||_2 \).

Give a formula for the reconstructed initial vector \( c^e(0) = A^{-1}_{t_f} \cdot c^e(t_f) \) and the thus made error \( ||c^e(0) - c(0)||_2 \). Explain why the problem requires regularization.

5. Let \( c^{e,\alpha}(0) \) denote the solution to the Tychonov regularized backwards problem

\[
 c^{e,\alpha}(0) = (A^T A + \alpha I)^{-1} A^T \cdot c^e(t_f)
\]

Compute the error to the correct initial condition \( ||c^{e,\alpha}(0) - c(0)||_2 \) in dependence on the regularization parameter \( \alpha \), and find the \( \hat{\alpha} \) which this error becomes minimal for. You can do this either by hand (doable, but technical) or by writing a matlab program, which runs through different values of \( \alpha \) and finds the minimizer. Plot the error as a function of \( \alpha \) for interesting values of \( N \) and \( \delta \).

6. For \( N = 5, 10, 15 \), produce plots of the optimal \( \hat{\alpha} \), which minimizes the error, in dependence on the relative error size \( \delta \). Again, this can be computed by hand, or in matlab, by running step 5 for a list of values for \( \delta \). Compare the results to the theoretical estimate for the optimal \( \alpha \), provided in Section 8.2 in Strang’s CSE book. Explain possible deviations.