Exercise 9

Consider the domain $\Omega \subset \mathbb{R}^2$, as given in the figure below. The boundary $\Gamma$ consists of Dirichlet boundaries $\Gamma_D$ (thin lines) and Neumann boundaries $\Gamma_N$ (fat lines). Using finite differences, we would like to approximate the Poisson problem

\[
\begin{align*}
-\Delta u &= 1 \quad \text{in } \Omega \\
 u &= f \quad \text{on } \Gamma_D \\
 \frac{\partial u}{\partial n} &= \frac{\partial f}{\partial n} \quad \text{on } \Gamma_N
\end{align*}
\]

where $f(x, y) = x^3y - xy^3 - \frac{1}{2}x^2$.

Write a Matlab function that, for any given resolution parameter, yields a linear system that discretizes the above Poisson problem.

Then choose a small resolution, a medium resolution and a high resolution, and (try to) solve the linear systems by

1. elimination with reordering
2. Jacobi iteration
3. a simple multigrid method (V-cycle)
4. conjugate gradients

Compare the approaches with respect to accuracy and run times. Up to which resolution can you go with the best method?

Remarks:

- Construct the matrix for interior points only, using lexicographic ordering. View $\Omega$ either as a composition or a subtraction of two rectangles. Rectangular geometries can be discretized using Matlab’s `kron`.
- For elimination, use Matlab functions. The other three methods have to be implemented by hand (codes from the CSE web page can be used).
- Be aware of the difference between discretization error and errors in solving the
linear system approximately.

- For a small test problem \( A\vec{u} = \vec{b} \), the accuracy of a linear solver is best measured by the norm of the error \( \|\vec{u}_{\text{approx}} - \vec{u}_{\text{correct}}\| \). However, for high resolutions, this error may not be accessible. In those cases, the residual \( \|A\vec{u}_{\text{approx}} - \vec{b}\| \) can be used.

- The high resolution may cause some solution methods to fail. The medium resolution shall be chosen, so that all considered solution methods succeed.