MIT Dept. of Mathematics
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Exercise Sheet 1

Out Wed 02/06/08 Due Fri 02/22/08

Exercise 1
The telegraph equation $u_{tt} + 2du_t = u_{xx}$ describes the evolution of a signal in an electrical transmission line (`en.wikipedia.org/wiki/Telegraph_equation`). Consider $x \in [-\pi, \pi)$ with periodic boundary conditions. Find the solution by a Fourier approach. Show that the waves e^{ikx} travel with frequency dependent velocities, while being damped with time.

Exercise 2
Write a program that approximates the solution of the beam equation $u_t = -u_{xxxx}$ on $x \in [-1, 1]$ by finite differences, given the boundary conditions $u(-1,t) = u(1,t) = u_x(-1,t) = u_x(1,t) = 0$. Plot the numerical solution $u(x,t)$ at times $t_1 = 0.03$ and $t_2 = 0.08$, when starting with initial condition $u(x,0) = (1-x^2)^2$. You can use the code `mit18086_fd_heateqn.m` on the course web page as a guideline.

Exercise 3
Use finite differences to solve Airy's equation $u_t = u_{xxx}$ on $x \in [-1, 1)$, using periodic boundary conditions, and starting with initial condition $u(x,0) = \cos(\pi x) + \cos(2\pi x)$.

1. Approximate u_{xxx} by a four point stencil. Investigate the four possible placements of the stencil (completely upwind, two points upwind–one point downwind, one point upwind–two points downwind, completely downwind) for stability, using von Neumann stability analysis. Implement the stable approximations in a Matlab program. Plot the numerical solution at times $t_1 = 0.01$ and $t_2 = 0.03$, using a reasonable grid resolution.
2. Derive a centered symmetric five point stencil that approximates u_{xxx}. Show that it leads to an unstable method. Recover stability by adding a numerical diffusion term cu_{xx}, or by adding a numerical super-diffusion term $-cu_{xxxx}$ (whatever you think works better). Use the Lax-Friedrichs approach, as seen in the lecture for $u_t = u_x$, as a guideline. Show that the new method is stable, given the constant c is chosen large enough. Implement your method (you can use your code from the previous part) and plot the function at times $t_1 = 0.01$ and $t_2 = 0.03$, for a suitably chosen value of c.