Simulating two-phase flow using a level set method

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1 Problem definition

1.1 NS equations

We consider the 2-D fluid motion for falling oil droplets in air, as the cases being studied in [1]. We denote the density and viscosity inside the droplet by ρ_l and μ_l, respectively, and for the gas phase by ρ_g and μ_g. From the conservation of momemtum we get the incompressible Navier-Stokes equations

$$u_t + (u \cdot \nabla)u = F + \tfrac{1}{\rho}(-\nabla\rho + \nabla \cdot (2\mu D) + \sigma k \delta(d)\mathbf{n}) \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

where $\mathbf{u} = (u, v)$ is the fluid velocity, $\rho = \rho(x,t)$ is the fluid density, $\mu = \mu(x,t)$ is the fluid viscosity, D is the viscous stress tensor, and $\mathbf{F}$ is a body force. The surface tension term is considered to be a force concentrated on the interface. σ is the surface tension, k is the curvature of the front, d is the normal distance to the front, δ is the Dirac delta function and $\mathbf{n}$ is the unit outward normal vector at the front. For immiscible liquids the density and viscosity are constant on particle paths, therefore

$$\rho_t + (\mathbf{u} \cdot \nabla)\rho = 0 \tag{3}$$

$$\mu_t + (\mathbf{u} \cdot \nabla)\mu = 0 \tag{4}$$

Free-slip condition $\mathbf{u} \cdot \mathbf{n} = 0$ is used for wall boundaries where $\mathbf{n}$ is the normal vector at the boundary.

If the intial radius of the drop is denoted as R and the only body force considered is gravity denoted as $\mathbf{g}$. We can get the non-dimensionalized form of Eq.(1)

$$\mathbf{u}_t = -(\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{g}_u + \frac{1}{\rho}\left(-\nabla p + \frac{1}{Re}\nabla \cdot (2\mu D) + \frac{1}{B} k\delta(d)\mathbf{n}\right) \tag{5}$$

The key parameters are ρ_l/ρ_g,density ratio;μ_l/μ_g, viscosity ratio; $Re = (2R)^{3/2}\sqrt{g}\rho_g/\mu_g$; and $B = 4\rho_g g R^2/\sigma$. The dimensionless density and viscosity outside the droplet are equal to 1.

1.2 Projection method

We correct the intermediate velocity field bu the gradient of a pressure P^{n+1} to enforce incompressibility. Hence we get

$$-\frac{\nabla P^{n+1}}{\rho^n} = \frac{1}{dt}\mathbf{U}^{n+1} - \frac{1}{dt}\mathbf{U}^n \tag{6}$$

To compute the projection, we take the divergence of both sides of the above equation to obtain

$$-\left(\nabla \cdot \left(\frac{\nabla P^{n+1}}{\rho^n}\right)\right) = -\frac{1}{dt}\nabla \cdot \mathbf{U^n} \tag{7}$$

After the linear system is solved, we can update the intermediate velocity field to the incompressible velocity by

$$\mathbf{U} = \mathbf{U} - dt\frac{\nabla P}{\rho} \tag{8}$$

1.3 Level set approach

We use a level set function ϕ to track the interface, and it is taken positive outside the droplet and negative inside the droplet. The ϕ is intialized to be the signed normal distance from the interface. Consider the following equation:

$$\phi_t + (\mathbf{u} \cdot \nabla)\phi = 0 \tag{9}$$

It should be noted that while ϕ is initially a distance function it will remain so (i.e., $|\nabla \phi| \neq 1$. Furthermore, solutions of ϕ can develop a jump at the interface when interfaces merge. So we need reinitializing step to reconstruct the distance function. This is achieved by solving the following problem to steady state

$$\phi_t = S(\phi_0)\left(1 - \sqrt{\phi_x^2 + \phi_y^2}\right) \tag{10}$$

$$\phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}) \tag{11}$$

where S is the sign function. The above equation has the property that ϕ remains unchanged at the interface; Away from the interface ϕ will converge to $|\nabla \phi| = 1$. For numerical purposes it is smoothed according to

$$S_\epsilon(\phi_0) = \frac{\phi_0}{\sqrt{\phi_0^2 + \epsilon^2}} \tag{12}$$

1.4 Algorithm

- Initialize $\phi(\mathbf{x}, 0)$ to be signed normal distance to the front.
- Solve

$$\mathbf{u}_t = \mathbf{P}_\rho(\mathbf{u}), \phi_t + (\mathbf{u} \cdot \nabla)\phi = 0 \tag{13}$$

for one time step. Denote the updated ϕ by $\phi^{(n+1/2)}$, and the updated velocity by $\mathbf{u}^{(n+1)}$.

- Reinitializing ϕ by solving $\phi_t = S(\phi^{(n+1/2)})(1 - |\nabla(\phi)|)$ with $\phi(\mathbf{x}, 0) = \phi^{(n+1/2)}(x)$ to steady state. Denote the steady state solution by $\phi^{(n+1)}$.
- Now complete one time step. The zero level set of $\phi^{(n+1)}$ gives the new interface position. Repeat steps 2 and 3.

1.5 Discretization

Discretization is almost in the same scheme with ref.[2]. The density and distance function are also defined in the center of a cell.

In computing the surface tension term, special care must be put into the δ function. One can smear the interface or using a smart ghost-fluid method. We cast the surface tension force in the level set formulation using a smoothed δ function.

$$(1/B)\kappa\delta(d)\mathbf{n} = (1/B)\kappa(\phi)\delta(\phi)\nabla\phi, \tag{14}$$

where the curvature is $\kappa(\phi) = \nabla \cdot (\nabla\phi/|\nabla\phi|)$ and the smoothed δ function is

$$\delta_\alpha(\phi) \equiv \begin{cases} \frac{1}{2}(1 + \cos(\pi\phi/\alpha))/\alpha & if|\phi| < \alpha \\ 0 & otherwise. \end{cases} . \tag{15}$$

where α is the thickness of the interface, and $\alpha = \frac{3}{2}\delta x$ in this simulation.

In the same sense, we smooth the density in the interface by the following

$$\begin{aligned} &\bar{\rho} = (\rho_b + \rho_c)/(2\rho_c) \\ &\Delta\rho = (\rho_c - \rho_b)/(2\rho_c) \\ &\rho(\phi) = \begin{cases} 1 & if \phi > \alpha \\ \rho_b/\rho_c & if \phi < -\alpha \\ \bar{\rho} + \Delta\rho \sin(\pi\phi/(2\alpha)) & otherwise. \end{cases} \end{aligned} \tag{16}$$

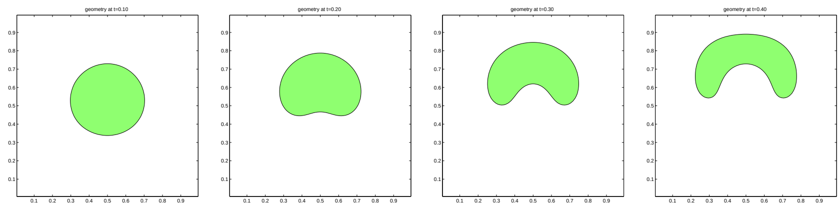
where ρ_c and ρ_b are the density of surroundings and droplets respectively. And we assume the same viscosity in the whole simulation.

2 Results

We consider a droplet/bubble in a square box. In all the cases, a non-slip boundary condition is used in all four walls.

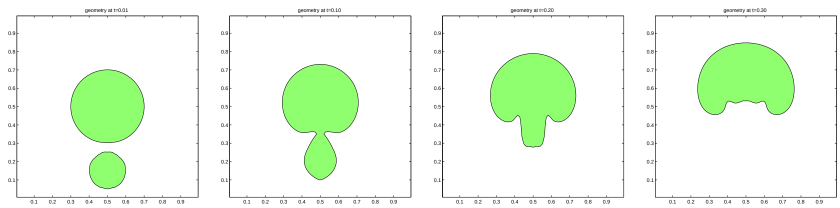
2.1 Bubble rising

Figure 1 shows a bubble rising in a fluid from rest. Note that a high density ratio up to 1000 is achieved. Figure 2 shows the evolution of two bubbles colliding with each other.



(a) (b) (c) (d)

Figure 1: *Bubble rising with medium range Reynolds number; Re=100, B=200, density ratio 1000/1, grid 90×90.*

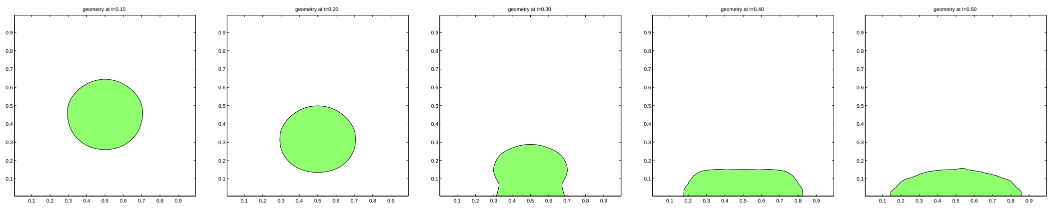


(a) (b) (c) (d)

Figure 2: *Bubble merge; Re=100, B=200, density ratio 1000/1, grid 90×90.*

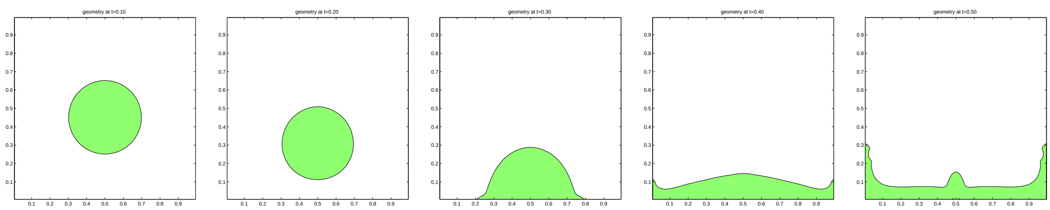
2.2 Droplet folling

We consider three cases. The first problem is a drop that is allowed to fall from rest and hit the bottom of our closed box, the second problem is the same condition without surface tension, the last problem simulate a drop very a very lower density ratio. Figure 3 show the evolution of a water drop with surface tension. Figure 4 show the evolution of a water drop without surface tension. Figure 5 shows the evolution of interface with a small density ratio and small surface tension. It can be seen that the shape of interface is preserved with a high density ratio and a high surface tension.



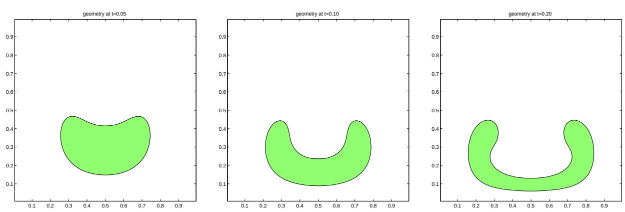
(a) (b) (c) (d) (e)

Figure 3: *Evolution of a water drop with surface tension. Drop remains circular as it hits the base; Re=10, B=1/800, density ratio 1/1000, grid 90×90.*



(a) (b) (c) (d) (e)

Figure 4: *Evolution of a water drop without surface tension. Re=10, B=Inf, density ratio 1/1000, grid 90×90.*

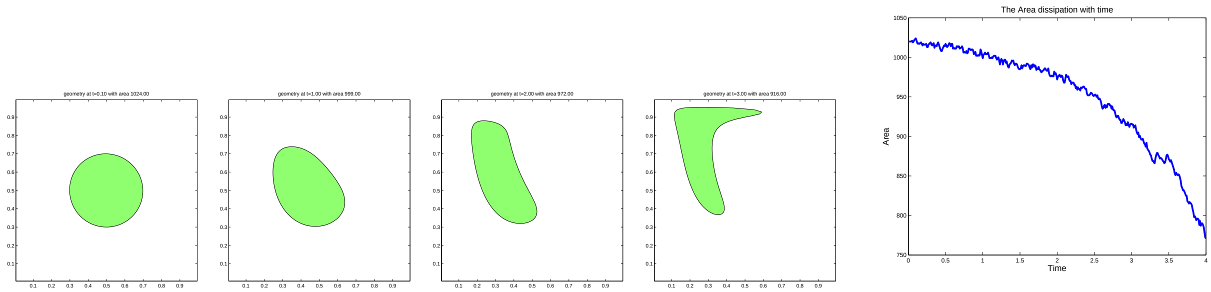


(a) (b) (c)

Figure 5: *Evolution of a droplet without surface tension. Re=10, B=Inf, density ratio 1/2, grid 90×90.*

2.3 Limitations

In order to study the limitations of current level set formulation, we let a initial circle evolving in a lid driven box, with no density differences and surface tension. Figure 6 shows the evolution of the circle plus the dissipation curve of the circular area. A high loss of area (i.e. mass) can be observed. This may due to the upwind scheme used. More investigation shows that smaller dt will introduce more loss of area.



(a) (b) (c) (d) (e)

Figure 6: *Evolution of a circular in a lid-driven box, grid 90×90.*

3 Conclusions

In summary, a level set method was used to track the interface of two incpmpressible fluids. The algorithm is easy to code since it uses a Eulerian mesh. By now, a density ratio up to 1000 can be simulated. In addition, the merging, breakup of the interface can be handled automatically. However, due to the upwind scheme used in evolving the level set equation, the area of the droplet/bubble doesn't conserved. Further investigation of high order scheme such as ENO may resolve this problem.

References

[1] Mark Sussman, Peter Smereka and Stanley Osher, *A Level Set Approach for Computing Solutions to Incompressible Two-Phase Flow*, J. Comp. Phys, 114, 146-159(1994).

[2] Benjamin Seibold, *A compact and fast Matlab code solving the incompressible Navier-Stokes equations on rectangular domains*, http://www-math.mit.edu/ seibold (2008).