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18.086 spring 2007
Exercise Sheet 1

Out Wed 02/07/07
Due Wed 02/21/07

Exercise 1 Consider the function $u(x,t)$ evolving under the initial value problems

$$\text{(i) } u_t = u_x \quad , \qquad \text{(ii) } u_t = u_{xx} \quad , \qquad \text{(iii) } u_t = u_{xxx} \quad , \qquad \text{(iv) } u_t = u_{xxxx}$$

with initial conditions $u(x,0) = u_0(x)$.

1. Consider $x \in [-\pi, \pi)$ with periodic boundary conditions. Use Fourier series to obtain the analytic solution to the four initial value problems (i)–(iv). How smooth do the initial conditions u_0 have to be? Use the Fourier representation to describe the behavior of the solutions (briefly). What would change in all four cases, if the right hand side had a minus sign?
2. Write a Matlab program which approximates the solution of $u_t = -u_{xxxx}$ on the domain $(x,t) \in [-1,1] \times [0,0.1]$ by finite differences, given the boundary conditions $u(-1,t) = u(1,t) = u_x(-1,t) = u_x(1,t) = 0$. Plot the function $u(x,t)$ at times $t_1 = 0.05$ and $t_2 = 0.1$, when starting with initial condition $u(x,0) = (1-x^2)^2$. You can use the code solving $u_t = u_{xx}$, given on the course webpage, as a guideline.
3. Consider Airy's equation $u_t = u_{xxx}$. Discretize the third space derivative by a four point stencil. Investigate the four possible placements of the stencil (completely upwind, two points upwind–one point downwind, one point upwind–two points downwind, completely downwind) for stability, using von Neumann stability analysis. Implement the stable approximations in a Matlab program, solving Airy's equation on $(x,t) \in [-1,1] \times [0,1]$, using appropriate boundary conditions of your choice. (Remark: If you do not use periodic boundary conditions, you have to find out how many conditions to prescribe on each side of the computational interval.) Plot the function at times $t_1 = 0.5$ and $t_2 = 1$.
4. Show that using a centered five point stencil to approximate the third derivative in Airy's equation yields an unstable method. Recover stability by a Lax-Friedrichs approach, as seen in the lecture for the one-way wave equation. (Remark: Here one can add cu_{xx}, as for $u_t = u_x$, but one can also add $-cu_{xxxx}$.) Show that the new method is stable. Implement this method (you can use your code from the previous part) and plot the function at times $t_1 = 0.5$ and $t_2 = 1$.