# 7.2 Calculus of Variations

One theme of this book is the relation of equations to minimum principles. ***To minimize $P$ is to solve $P' = 0$.*** There may be more to it, but that is the main point. For a quadratic $P(u) = \frac{1}{2}u^{\mathrm{T}}Ku - u^{\mathrm{T}}f$, there is no difficulty in reaching $P' = Ku - f = 0$. The matrix $K$ is symmetric positive definite at a minimum.

In a continuous problem, the "derivative" of $P$ is not so easy to find. The unknown $u(x)$ is a function, and $P(u)$ is usually an integral. Its derivative $\delta P/\delta u$ is called the ***first variation***. ***The "Euler-Lagrange equation" $\delta P/\delta u = 0$ has a weak form and a strong form***. For an elastic bar, $P$ is the integral of $\frac{1}{2}c(u'(x))^2 - f(x)u(x)$. The equation $\delta P/\delta u = 0$ is linear and the problem will have boundary conditions:

**Weak form** $\displaystyle\int cu'v'\,dx = \int fv\,dx$ for every $v$ **Strong form** $-(cu')' = f(x)$.

Our goal in this section is to get beyond this first example of $\delta P/\delta u$.

The basic idea should be simple and it is: ***Perturb $u(x)$ by a test function $v(x)$***. Comparing $P(u)$ with $P(u+v)$, the linear term in the difference yields $\delta P/\delta u$. *This linear term must be zero for every admissible $v$ (weak form).* This program carries ordinary calculus into the calculus of variations. We do it in several steps:

**1.** One-dimensional problems $P(u) = \int F(u, u')\,dx$, not necessarily quadratic

**2.** Constraints, not necessarily linear, with their Lagrange multipliers

**3.** Two-dimensional problems $P(u) = \iint F(u, u_x, u_y)\,dx\,dy$

**4.** Time-dependent equations in which $u' = du/dt$.

At each step the examples will be as familiar (and famous) as possible. In two dimensions that means Laplace's equation, and minimal surfaces in the nonlinear case. In time-dependent problems it means Newton's Laws, and relativity in the nonlinear case. In one dimension we rediscover the straight line and the circle.

This section is also the opening to ***control theory***—the modern form of the calculus of variations. Its constraints are differential equations, and Pontryagin's maximum principle yields solutions. That is a whole world of good mathematics.

***Remark*** To go from the strong form to the weak form, *multiply by $v$ and integrate*. For matrices the strong form is $A^{\mathrm{T}}CAu = f$. The weak form is $v^{\mathrm{T}}A^{\mathrm{T}}CAu = v^{\mathrm{T}}f$ for all $v$.

For functions with $Au = u'$, this exactly matches $\int cu'v'\,dx = \int fv\,dx$ above.

## One-dimensional Problems

The basic problem is to minimize $P(u)$ with a boundary condition at each end:

$$P(u) = \int_0^1 F(u, u')\,dx \quad \text{with} \quad u(0) = a \quad \text{and} \quad u(1) = b\,.$$

The best $u$ defeats every other candidate $u+v$ that satisfies these boundary conditions. Then $(u+v)(0) = a$ and $(u+v)(1) = b$ require that $v(0) = v(1) = 0$. When $v$ and $v'$ are small the correction terms come from $\partial F/\partial u$ and $\partial F/\partial u'$. They don't involve $v^2$:

**Inside the integral** $\quad F(u+v, u'+v') = F(u, u') + v\dfrac{\partial F}{\partial u} + v'\dfrac{\partial F}{\partial u'} + \cdots$

**After integrating** $\quad P(u+v) = P(u) + \displaystyle\int_0^1 \left( v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'} \right) dx + \cdots$

That integrated term is the "first variation". We have already reached $\delta P/\delta u$:

$$\textbf{Weak form} \qquad \frac{\delta P}{\delta u} = \int_0^1 \left( v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'} \right) dx = 0 \quad \text{for every } v\,. \tag{1}$$

This is the equation for $u$. The derivative of $P$ in each direction $v$ must be zero. Otherwise we can make $\delta P/\delta u$ negative, which would mean $P(u+v) < P(u)$: *bad.*

The strong form looks for a single derivative which—if it is zero—makes all these directional derivatives zero. It comes from integrating $\delta P/\delta u$ by parts:

$$\textbf{Weak form / by parts} \quad \int_0^1 \left( v\,\frac{\partial F}{\partial u} - v\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right) dx + \left[ v\,\frac{\partial F}{\partial u'} \right]_0^1 = 0\,.$$

The boundary term vanishes because $v(0) = v(1) = 0$. To guarantee zero for *every* $v(x)$ in the integral, the function multiplying $v$ must be zero (**strong form**):

$$\textbf{Euler-Lagrange equation for } \boldsymbol{u} \qquad \boxed{\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0\,.} \tag{2}$$

**Example** Find the shortest path $u(x)$ between two points $(0, a)$ and $(1, b)$.

By Pythagoras, $\sqrt{(dx)^2 + (du)^2}$ is a short step on the path. So $P(u') = \int \sqrt{1 + (u')^2}\,dx$ is the length of the path between the points. This square root $F(u')$ depends only on $u'$ and $\partial F/\partial u = 0$. The derivative $\partial F/\partial u'$ brings the square root into the denominator:

$$\textbf{Weak form} \quad \frac{\partial F}{\partial u} = \int_0^1 v'\frac{u'}{\sqrt{1 + (u')^2}}\,dx = 0 \quad \text{for every } v \text{ with } v(0) = v(1) = 0\,. \tag{3}$$

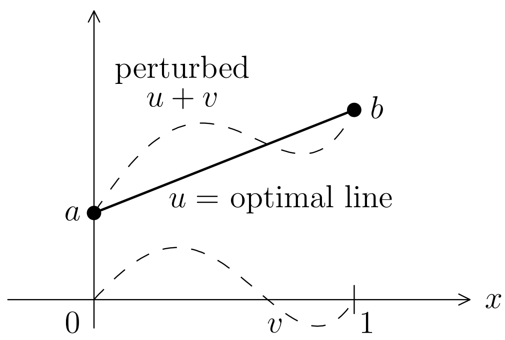
If the quantity multiplying $v'$ is a constant, then (3) is satisfied. The integral is certain to be zero because $v(0) = v(1) = 0$. The strong form *forces* $\partial F/\partial u'$ to be constant: the Euler-Lagrange equation (2) is

$$-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = -\frac{d}{dx}\frac{u'}{\sqrt{1+(u')^2}} = 0 \quad \text{or} \quad \frac{u'}{\sqrt{1+(u')^2}} = c\,. \tag{4}$$

***That integration is always possible when $F$ depends only on $u'$*** ($\partial F/\partial u = 0$). It leaves the equation $\partial F/\partial u' = c$. Squaring both sides, $u$ is seen to be *linear*:

$$(u')^2 = c^2(1+(u')2) \quad \text{and} \quad u' = \frac{c}{\sqrt{1-c^2}} \quad \text{and} \quad u = \frac{c}{\sqrt{1-c^2}}x + d\,. \tag{5}$$

The constants $c$ and $d$ are chosen to match $u(0) = a$ and $u(1) = b$. *The shortest curve connecting two points is a straight line.* No surprise! The length $P(u)$ is a minimum, not a maximum or a saddle point, because the second derivative $F''$ is positive.




typical perturbation $v(x)$ area below arc is $A$

Figure 7.5: Shortest paths from $a$ to $b$: straight line and circular arc (constrained).

## Constrained Problems

Suppose we cannot go in a straight line because of a constraint. When the constraint is $\int u(x)\,dx = A$, we look for ***the shortest curve that has area*** $A$ ***below it***:

$$\text{Minimize} \quad P(u) = \int_0^1 \sqrt{1+(u')^2}\,dx \quad \text{with} \quad u(0)=a,\ u(1)=b,\ \int_0^1 u(x)\,dx = A\,.$$

The area constraint should be built into $P$ by a Lagrange multiplier—here called $m$. The multiplier is a number and not a function, because there is one overall constraint rather than a constraint at every point. The Lagrangian$L$ builds in $\int u\,dx = A$:

**Lagrangian** $L(u,m) = P + (\text{multiplier})(\text{constraint}) = \int (F + mu)\,dx - mA\,.$

The Euler-Lagrange equation $\delta L/\delta u = 0$ is exactly like $\delta P/\delta u = 0$ in (2):

$$\frac{\partial(F+mu)}{\partial u} - \frac{d}{dx}\left[\frac{\partial(F+mu)}{\partial u'}\right] = m - \frac{d}{dx}\frac{u'}{\sqrt{1+(u')^2}} = 0\,. \tag{6}$$

Again this equation is favorable enough to be integrated:

$$mx - \frac{u'}{\sqrt{1+(u')^2}} = c \quad \text{which gives} \quad u' = \frac{mx-c}{\sqrt{1-(mx-c)^2}}\,.$$

After one more integration we reach the equation of a circle in the $x$-$u$ plane:

$$u(x) = \frac{-1}{m}\sqrt{1-(mx-c)^2} + d \quad \text{and} \quad (mx-c)^2 + (mu-d)^2 = 1\,. \tag{7}$$

***The shortest path is a circular arc***! It goes high enough to enclose area $A$. The three numbers $m, c, d$ are determined by the conditions $u(0) = a$, $u(1) = b$, and $\int u\,dx = A$. The arc is drawn in Figure 7.5 (and $m$ is negative).

We now summarize the one-dimensional case, allowing $F$ to depend also on $u''$. That introduces $v''$ into the weak form and needs *two* integrations by parts to reach the Euler-Lagrange equation. When $F$ involves a varying coefficient $c(x)$, the form of the equation does not change, because it is $u$ and not $x$ that is perturbed.

The first variation of $\boldsymbol{P(u) = \int_0^1 F(u, u', u'', x)\,dx}$ is zero at a minimum:

**Weak form** $$\frac{\delta P}{\delta u} = \int_0^1 \left( v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'} + v''\frac{\partial F}{\partial u''}\right) dx = 0 \text{ for all } v.$$

The Euler-Lagrange equation from integration by parts determines $\boldsymbol{u(x)}$:

**Strong form** $$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right) = 0\,.$$

Constraints on $\boldsymbol{u}$ bring Lagrange multipliers and saddle points of $\boldsymbol{L}$.

Applications are everywhere, and we mention one (of many) in sports. What angle is optimal in shooting a basketball? The force of the shot depends on the launch angle—line drives or sky hooks need the most push. The force is minimized at 45° if the ball leaves your hand ten feet up; for shorter people the angle is about 50°. What is interesting is that the same angle solves a second optimization problem: to have the largest margin of error and still go through the hoop.

The condition is $P' = 0$ in basketball (one shot) and $\delta P/\delta u = 0$ in track—where the strategy to minimize the time $P(u)$ has been analyzed for every distance.

## Two-dimensional Problems

In two dimensions the principle is the same. The starting point is a quadratic $P(u)$, without constraints, representing the potential energy over a plane region $S$:

**Minimize** $$P(u) = \iint_S \left[\frac{c}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{c}{2}\left(\frac{\partial u}{\partial y}\right)^2 - f(x,y)\,u(x,y)\right] dx\,dy\,.$$

If this energy has its minimum at $u(x,y)$, then $P(u+v) \geq P(u)$ for every $v(x,y)$. We mentally substitute $u+v$ in place of $u$, and look for the term that is linear in $v$. That term is the first variation $\delta P/\delta u$, which must be zero for every $v(x,y)$:

**Weak form**
$$\frac{\delta P}{\delta u} = \iint_S \left[ c\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} + c\,\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y} - fv \right] dx\,dy = 0\,. \tag{8}$$

This is the **equation of virtual work**. It holds for all admissible functions $v(x,y)$, and it is the weak form of Euler-Lagrange. The strong form requires as always an integration by parts (Green's formula), in which the boundary conditions take care of the boundary terms. Inside $S$, that integration moves derivatives away from $v(x,y)$:

**Integrate by parts**
$$\iint_S \left[ -\frac{\partial}{\partial x}\left(c\,\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\,\frac{\partial u}{\partial y}\right) - f \right] v\,dx\,dy = 0\,. \tag{9}$$

Now the strong form appears. This integral is zero for every $v(x,y)$. By the "fundamental lemma" of the calculus of variations, the term in brackets is forced to be zero everywhere:

**Strong form**
$$-\frac{\partial}{\partial x}\left(c\,\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\,\frac{\partial u}{\partial y}\right) = f(x,y) \quad \text{throughout } S\,. \tag{10}$$

This is the Euler-Lagrange equation $A^{\mathrm{T}}CA = f$, or $-\nabla \cdot c\nabla u = f$. For constant $c$ it is Poisson. If the $y$ variable is removed, we are back to a one-dimensional rod.

With no extra effort we can go backwards to $P(u)$ from any linear equation:

**Second-order equation**
$$a\,\frac{\partial^2 u}{\partial x^2} + 2b\,\frac{\partial^2 u}{\partial x \partial y} + c\,\frac{\partial^2 u}{\partial y^2} = 0\,. \tag{11}$$

When $a, b$, and $c$ are constant, the corresponding quadratic "energy" is

$$P(u) = \frac{1}{2}\iint \left[ a\left(\frac{\partial u}{\partial x}\right)^2 + 2b\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + c\left(\frac{\partial u}{\partial y}\right)^2 \right] dx\,dy\,.$$

If we minimize $P$ we expect to reach (11) as its Euler equation. But there is more to it than that. To *minimize* $P$ it should be *positive definite*. Inside the integral is an ordinary 2 by 2 quadratic $au_x^2 + 2bu_xu_y + cu_y^2$. The test for positive-definiteness is still $ac > b^2$, as it was in Chapter 1. (We can make $a > 0$ in advance.) That test decides whether or not equation (11) can be solved with arbitrary boundary values on $u(x,y)$.

In this positive definite case the equation is called "elliptic" and minimization is justified. There are three fundamental classes of partial differential equations:

The partial differential equation $au_{xx} + 2bu_{xy} + cu_{yy} = 0$ is **elliptic** or **parabolic** or **hyperbolic**, according to the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$:

| | | | | |
|---|---|---|---|---|
| **E** | $ac > b^2$ | elliptic boundary-value problems | **(steady state equations)** |
| **P** | $ac = b^2$ | parabolic initial-value problems | **(heat/diffusion equations)** |
| **H** | $ac < b^2$ | hyperbolic initial-value problems | **(wave/convection equations)** |

Laplace's equation $u_{xx} + u_{yy} = 0$ is elliptic, with $a = c = 1$ producing the identity matrix. The heat equation $u_{xx} - u_t = 0$ is parabolic, with $b = c = 0$; the matrix is singular and its determinant is zero. That parabolic borderline between elliptic and hyperbolic needs help from lower-order terms. The wave equation $u_{xx} - u_{tt} = 0$ has $a = 1$ and $c = -1$. It asks for initial values—two conditions on part of the boundary and no conditions on another part, instead of one condition everywhere.

Here we stay with elliptic equations and minima of $P(u)$. ***The boundary conditions can specify*** $u$, ***or they can specify the normal component*** $w \cdot n$. Comparing (8) with (9), the boundary term comes from Green's formula in Section ____:

$$\iint_S c\nabla u \cdot \nabla v \, dx\, dy = -\iint_S (\nabla \cdot (c\nabla u)) v \, dx\, dy + \int_C (c\nabla u \cdot n) v \, ds \,. \qquad (12)$$

Both sides equal $\iint f v \, dx\, dy$; that is the weak form. The first term on the right yields the strong form $-\operatorname{div}(c \operatorname{grad} u) = f(x, y)$. A zero integral over the boundary $c$ in (12) will be achieved by the boundary conditions. Strictly speaking, the boundary conditions on $u(x, y)$ are part of the strong form.

There are two ways to make this boundary integral of $(c\nabla u \cdot n)v$ safely zero. If boundary values $u = u_0$ are given, and $u + v$ shares those values, then $v = 0$ on the boundary. That kills the integral. When $u$ is not given and $v$ is free, we must impose the natural boundary condition $c\nabla u \cdot n = w \cdot n = 0$.

A natural condition on $w$ goes with $A^{\mathrm{T}}$. The essential condition $u = u_0$ goes with $A$.

## The Minimal Surface Problem

Now we are ready for ***nonlinear*** partial differential equations. The corresponding energy will not be a quadratic $P(u)$. It will be the exact energy $E(u)$, from which $P$ originally came as an approximation. If there is a thin membrane covering the set $S$—like a soap bubble—then stretching this membrane requires energy. The energy is proportional to the ***surface area of the soap bubble***. In the right units the material constant will be $c = 1$, and the problem is to ***minimize the surface***

***area*** $E(u)$:

$$\textbf{Choose } \boldsymbol{u(x,y)} \textbf{ to minimize } E(u) = \iint_S \left[1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]^{1/2} dx\,dy\,. \quad (13)$$

When $u = 0$, the bubble is flat. The expression in brackets reduces to 1. That is the minimum if $u = 0$ is admissible. But suppose the bubble is created on a piece of wire that goes around $S$ at a varying height $u_0(x, y)$. The bubble has to stick to the wire, so the trivial solution $u = 0$ is not allowed. The bent wire imposes a boundary condition $u = u_0(x, y)$ at the edge of $S$, and the ***minimal surface problem*** is to find the smallest area $E(u)$. Physically, surface tension makes the area a minimum.

The test for a minimum is still $E(u) \le E(u + v)$. To compute the term $\delta E/\delta u$ that is linear in $v$, look at the part $A$ from $u$ alone, and the correction $B$ involving $v$:

$$A = 1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \quad \text{and} \quad B = 2\,\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 2\,\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + O(v^2)\,.$$

For small $v$, the square root is $\sqrt{A + B} = \sqrt{A} + B/2\sqrt{A} + \cdots$. Integrate both sides:

$$E(u + v) = E(u) + \iint_S \frac{1}{\sqrt{A}}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dx\,dy + \cdots\,. \quad (14)$$

Now $\delta E/\delta u$ is exposed. It is the integral in (14) and it is zero. That is the weak form of the minimal surface equation. Because of the square root of $A$, it is not linear in $u$. (It is always linear in $v$; that is the whole point of the first variation!) Integrating by parts to remove the derivatives from $v$ produces the Euler equation in its strong form:

**Minimal surface equation**
$$-\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{A}}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{A}}\frac{\partial u}{\partial y}\right) = 0\,. \quad (15)$$

This is not easy to solve, because of the square root in the denominator. For nearly flat bubbles, linearization approximates $\sqrt{A}$ by 1. *The result is Laplace's equation.* Perhaps it is only natural that the most important nonlinear equation in geometry should reduce to the most important linear equation, but still it is beautiful.

## Nonlinear Equations

The shortest distance and minimal surface problems are typical of the general case. The variational problem starts with an integral $E = \iint F\,dx\,dy$. Then $F$ depends on $x$ and $y$ and $u$ and derivatives like $\partial u/\partial x$:

$$F = F(x, y, u, D_1u, D_2u, \ldots)\,.$$

For an elastic bar there was only $D_1u = \partial u/\partial x$. For a soap bubble there is also $D_2u = \partial u/\partial y$. Higher derivatives are allowed, and we can think of $u$ itself as $D_0u$.

The comparison of $E(u)$ with the nearby $E(u+v)$ starts from an ordinary expansion like $F(u+v) = F(u) + F'(u)v + O(v^2)$. When $F$ depends on several derivatives of $u$, this expansion has more terms from $F(x, y, D_0 u + D_0 v, D_1 u + D_1 v, \ldots)$:

**Inside the integral** $$F(u+v) = F(u) + \sum \frac{\partial F}{\partial D_i u} D_i v + \cdots . \tag{16}$$

***We take the derivatives of $F$ with respect to $u$ and $u_x$ and any other $D_i u$.***

The weak form involving $v$ deals with those linear terms: (integral$=0$ for all $v$). The strong form lifts each derivative $D_i$ from $v$ and puts it (as $D_i^{\mathrm{T}}$) onto the part involving $u$:

**Weak to Strong** $$\iint \left(\frac{\partial F}{\partial D_i u}\right)(D_i v) dx\, dy \longrightarrow \iint \left[D_i^{\mathrm{T}}\left(\frac{\partial F}{\partial D_i u}\right)\right] v\, dx\, dy$$

The transpose is $D_i^{\mathrm{T}} = -D_i$ for derivatives of odd order (with an odd number of integrations by parts and minus signs). For even derivatives $D_i^{\mathrm{T}} = +D_i$.

Buried inside the calculus of variations is the real source of $A^{\mathrm{T}}CA$! The derivatives $D_i u$ give $Au$. Their "transposes" $D_i^{\mathrm{T}}$ give $A^{\mathrm{T}}$. In between, $C$ *is normally nonlinear*. But when $F$ is a pure quadratic $\frac{1}{2}c(Du)^2$, then $D^{\mathrm{T}}\partial F/\partial Du$ becomes $D^{\mathrm{T}}(cDu)$—which is exactly the linear $A^{\mathrm{T}}CAu$ the we know so well.

**7G** Each problem in the calculus of variations can be stated in three forms:

**Variational form** Minimize $$E(u) = \iint_S F(x, y, u, D_1 u, D_2 u, \ldots)\, dx\, dy$$

**Weak form** $$\frac{\delta E}{\delta u} = \iint_S \left(\sum \frac{\partial F}{\partial D_i u}\right)(D_i v)\, dx\, dy = 0 \text{ for all } v$$

**Euler-Lagrange strong form** $$\sum D_i^{\mathrm{T}}\left(\frac{\partial F}{\partial D_i u}\right) = 0.$$

**Example 1** $F = u^2 + u_x^2 + u_y^2 + u_{xx}^2 + u_{xy}^2 + u_{yy}^2 = (D_0 u)^2 + \cdots + (D_5 u)^2$

The derivatives of $F$ (a pure quadratic) are $2u, 2u_x, 2u_y, \ldots, 2u_{yy}$. They are derivatives with respect to $u$ and $u_x$ and the other $D_i u$, *not* with respect to $x$!

**Weak form** $$2\iint [uv + u_x v_x + u_y v_y + u_{xx} v_{xx} + u_{xy} v_{xy} + u_{yy} v_{yy}]\, dx\, dy = 0\,.$$

We integrate every term by parts to give the strong form (times $v$):

**Strong form** $$2\,[u - u_{xx} - u_{yy} + u_{xxxx} + u_{xyxy} + u_{yyyy}] = 0\,.$$

This is linear because $F$ is quadratic; the minus signs come with odd derivatives in $F$.

**Example 2** $F = (1 + u_x^2)^{1/2}$ or $F = (1 + u_x^2 + u_y^2)^{1/2}$

The derivatives with respect to $u_x$ and $u_y$ bring the square root into the denominator. The shortest path equation and the minimal surface equation are

$$-\frac{d}{dx}\frac{u_x}{(1+u_x^2)^{1/2}} = 0 \quad \text{and} \quad -\frac{\partial}{\partial x}\left(\frac{u_x}{F}\right) - \frac{\partial}{\partial y}\left(\frac{u_y}{F}\right) = 0\,.$$

Every term fits into the pattern of $A^{\mathrm{T}}CA$, and **the framework becomes nonlinear**:

$$e = Au \underset{\text{nonlinear}}{\longrightarrow} w = C(e) = \frac{\partial F}{\partial e} \quad \text{leads to} \quad A^{\mathrm{T}}w = A^{\mathrm{T}}C(Au) = f\,.$$

## Nonlinear $C(Au)$ from Nonquadratic Energies

That last line was worth a chapter of words. It is the shortest path to nonlinear equilibrium equations. In a linear spring, $w = ce$ is proportional to $e$. The internal strain energy is $F = \int ce\,de = \frac{1}{2}ce^2$. ***In a nonlinear spring the constitutive law is*** $w = C(e)$. The relation of force to stretching, or current to voltage, or flow to pressure, is no longer a straight line $Ce$. We need parentheses in $C(e)$!

The energy density is still $F = \int C(e)\,de$ = force times movement. Minimizing the total energy (integrate $F$ to find $E$) still gives the equilibrium equation:

$$\textbf{The energy} \quad P(u) = \int [F(Au) - fu]\,dx \quad \textbf{is minimized when} \quad A^{\mathrm{T}}C(Au) = f\,.$$

The first variation of $E$ leads to $\int [C(Au)(Av) - fv]\,dx = 0$ for every $v$ (weak form). Then $A^{\mathrm{T}}C(Au) = f$ is the Euler equation (strong form).

For the nonlinear equivalent of positive definiteness, *the function* $C(e)$ *should be increasing*. The line $w = ce$ had a constant slope $c > 0$. Now that slope $C' = dC/de$ is changing—but it is still positive. That makes the energy $E(u)$ a convex function. The Euler equation is elliptic—we have a minimum.

An example is the *power law* $w = C(e) = e^{p-1}$ with $p > 1$. The energy density is its integral $F = e^p/p$. The stretching is $e = Au = du/dx$. The equilibrium equation is $A^{\mathrm{T}}C(Au) = (-d/dx)(du/dx)^{p-1} = f$. This is linear for $p = 2$. Otherwise nonlinear.

## Complementary Energy

The complementary energy is a function of $w$ instead of $e$. It starts with the *inverse constitutive law* $e = C^{-1}(w)$—in our example $e = w^{1/(p-1)}$. The strain $e$ comes from the stress $w$; the arrow in our framework is reversed. Graphically, we are looking at Figure 7.6a from the side. ***The area under that curve is the complementary energy density*** $F^*(w) = \int C^{-1}(w)\,dw$. The twin equations come from $F$ and $F^*$:

$$\textbf{Constitutive Laws} \qquad w = C(e) = \frac{\partial F}{\partial e} \quad \text{and} \quad e = C^{-1}(w) = \frac{\partial F^*}{\partial w}\,. \tag{17}$$

The symmetry is perfect and the dual minimum principle applies to $Q(w) = \int F^*(w)\,dx$:

**The complementary energy** $Q(w)$ **is a minimum subject to** $A^{\mathrm{T}}w = f(x)$.

A Lagrange multiplier $u(x)$ takes $Q$ to $L(w,u) = \int [F^*(w) - uA^{\mathrm{T}}w + uf]\,dx$, with the constraint $A^{\mathrm{T}}w = f$ built in. Its derivatives recover the two equations of equilibrium, now nonlinear:

$$\begin{aligned} \partial L/\partial w = 0 \quad &\text{is} \quad C^{-1}(w) - Au = 0 \\ \partial L/\partial u = 0 \quad &\text{is} \quad A^{\mathrm{T}}w \qquad\quad = f\,. \end{aligned}$$

The first equation gives $w = C(Au)$ and then the second is $A^{\mathrm{T}}C(Au) = f$.

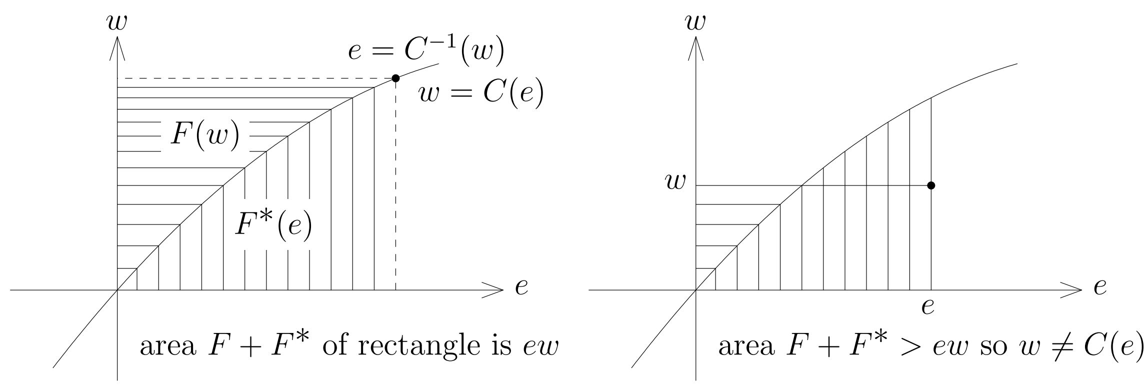



Figure 7.6: The graphs of $w = C(e)$ and $e = C^{-1}(w)$, and their areas $F^*$ and $F$.

Since these nonlinear things are in front of us, why not take the last step? It is never seen in advanced calculus, but there is nothing so incredibly difficult. It is the direct link between $F$ and $F^*$, known as the ***Legendre-Fenchel transform***:

$$F^*(w) = \max_e\,[ew - F(e)] \quad \text{and} \quad F(e) = \max_w\,[ew - F^*(w)]\,. \tag{18}$$

For the first maximum, differentiate with respect to $e$. That brings back $w = \partial F/\partial e$, which is the correct $C(e)$. The maximum itself is $F^* = e\partial F/\partial e - F$. The figures show graphically that the areas satisfy $F^* = ew - F$ on the curve and $ew - F < F^*$ off the curve. So the maximum of $ew - F$ is $F^*$ as desired.

Similarly, the second maximum in (18) leads to $e = \partial F^*/\partial w$. That is the constitutive law in the other direction, $e = C^{-1}(w)$. The whole nonlinear theory is there, provided the material laws are conservative—the energy in the system should be constant. This conservation law seems to be destroyed by dissipation, or more spectacularly by fission, but in some ultimate picture of the universe it must remain true.

***$F$ will be the Lagrangian and $F^*$ is the Hamiltonian***. The equations $mu_{tt} + cu_t + ku = 0$ and $u_{xx} = u_t$ include friction and damping and diffusion, but we hesitate to stretch our framework that far. Feynman's wonderful lectures made "least action" the starting point for physics, and the next paragraphs.

The Legendre transform reappears at full strength in constrained optimization. There $F$ and $F^*$ are more general convex functions (with nonnegative second derivatives) and ***we recognize that*** $F^{**}$ ***is*** $F$. Here we compute $F^*(w)$ for the power law and verify that it agrees with the integral of $C^{-1}(w)$.

**Example** Find $F^*(w)$ for the power law $F(e) = e^p/p$ ($e > 0$ and $w > 0$ and $p > 1$)
Differentiating $ew - F(e)$ gives $w - e^{p-1} = 0$. Then the conjugate $F^*(w)$ in (18) is $w^q/q$:

$$F^* = ew - \frac{1}{p}e^p = w^{1/(p-1)}w - \frac{1}{p}w^{p/(p-1)} = \frac{p-1}{p}w^{p/(p-1)} = \frac{1}{q}w^q\,.$$

$F^*(w) = w^q/q$ is also a power law, with *dual exponent* $q = p/(p-1)$. This matches the area under $C^{-1}(w) = w^{1/(p-1)}$, because integration will increase that exponent to $1 + 1/(p-1) = q$. The symmetric relation between the powers is $p^{-1} + q^{-1} = 1$.

## Dynamics and Least Action

Fortunately or unfortunately, the world is not in equilibrium. The energy stored in springs and beams and nuclei and people is waiting to be released. When the external forces change, the equilibrium is destroyed. Potential energy is converted to kinetic energy, the system becomes dynamic, and it may or may not find a new steady state.

When the system is conservative, the transients will not grow or decay. The energy changes from potential to kinetic to potential to kinetic, but the total energy $K+P$ remains constant. It is like the earth around the sun or a child on a frictionless swing. The force $dP/du$ is no longer zero, and the system oscillates. The problems are dynamic instead of static.

To describe the motion we need an equation or a variational principle. Numerically we mostly work with equations (Newton's laws and conservation laws). This section derives those laws from the ***principle of least action***:

The actual path $u(t)$ minimizes the action integral $A(u)$ between the initial state $u(t_0)$ and the final state $u(t_1)$:

$$A(u) = \int_{t_0}^{t_1} (\textbf{kinetic energy} - \textbf{potential energy})\,dt\,.$$

It is better to claim only that $\delta A/\delta u = 0$—the path is always a stationary point but not in every circumstance a minimum. We have a difference of energies, and positive definiteness can be lost (a saddle point). Laplace's equation will be overtaken by the wave equation. First come three examples to show how the ***global law*** of least action (the variational principle of least action) produces Newton's ***local law*** $F = ma$.

**Example 3** A ball of mass $m$ is attracted by the Earth's gravity

The only degree of freedom is the ball's height $u(t)$. The energies are $K$ and $P$:

$$K = \textbf{kinetic energy} = \frac{1}{2}m\left(\frac{du}{dt}\right)^2 \quad \text{and} \quad P = \textbf{potential energy} = mgu\,.$$

The action is $A = \int(\frac{1}{2}m(u')^2 - mgu)\,dt$. Then $\delta A/\delta u$ follows from the rules of this section—with the time variable $t$ replacing the space variable $x$. The true path $u$ is compared to its neighbors $u + v$. The linear part of $A(u + v) - A(u)$ gives the first variation $\delta A/\delta u = 0$:

$$\frac{\delta A}{\delta u} = \int_{t_0}^{t_1} (mu'v' - mgv)\,dt = 0 \quad \text{for every } v\,.$$

That is the weak form of Newton's law. You recognize the momentum $mu'$ as the derivative of $\frac{1}{2}m(u')^2$ with respect to the velocity $u'$. The strong form is the Euler equation:

$$\textbf{Newton's law} \qquad -\frac{d}{dt}\left(m\frac{du}{dt}\right) - mg = 0 \quad \text{which is} \quad ma = F\,. \tag{19}$$

**The action integral is minimized by the path that follows Newton's law.**

The 3-step framework of applied mathematics is not changed. The place of $A$ is taken by $d/dt$ and $A^{\mathrm{T}}$ is $-d/dt$. The material constant is $m$ and the external force is $f = mg$. The balance between $A^{\mathrm{T}}CAu$ and $f$ is Newton's law—a balance of **inertial forces** instead of mechanical forces. Figure 7.7 identifies the variables as the velocity and momentum.

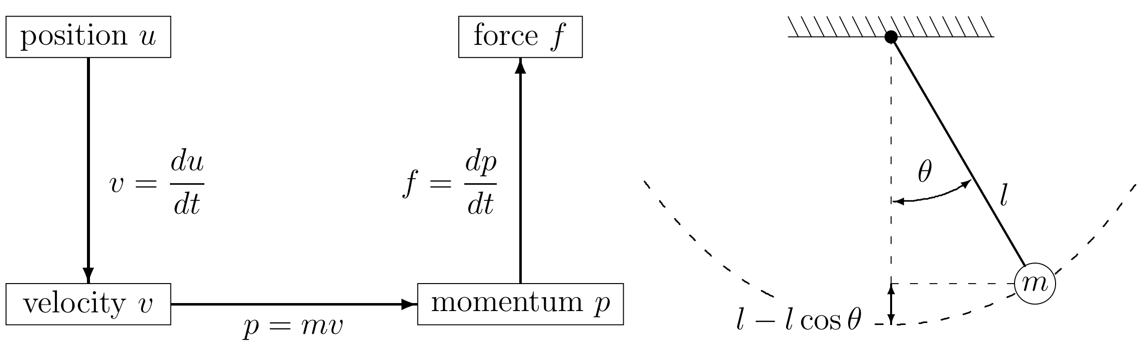



Figure 7.7: Newton's law for a ball. A pendulum also fits the framework.

**Example 4** A simple pendulum with mass $m$ and length $l$

The state variable $u$ is the angle $\theta$ from the vertical. The height $l - l\cos\theta$ still enters the potential energy and the velocity is $v = l\,d\theta/dt$:

$$K = \textbf{kinetic energy} = \frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 \quad \text{and} \quad P = \textbf{potential energy} = mg(l - l\cos\theta)$$

In this problem the equation will no longer be linear. $K$ is quadratic but $P$ involves $\cos\theta$. The Euler equation follows the rule for an integral $\int F(\theta, \theta')\,dt$, with $F = K - P$:

**Euler equation** $$\frac{\partial F}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial F}{\partial \theta'}\right) = 0 \quad \text{or} \quad -mgl\sin\theta - \frac{d}{dt}\left(ml^2\frac{d\theta}{dt}\right) = 0\,.$$

This is the equation of a simple pendulum. The mass cancels out; clocks keep time!

**Pendulum equation** $$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0\,. \tag{20}$$

When the angle is small and $\sin\theta$ is approximated by $\theta$, the equation becomes linear. The period changes a little. A linear clock keeps time, but not the right time.

**Example 5** A vertical line of springs and masses has $Mu'' + Ku = 0$

The potential energy in the springs is $P = \frac{1}{2}u^{\mathrm{T}}A^{\mathrm{T}}CAu$. The kinetic energy has $\frac{1}{2}m_i(du_i/dt)^2$ from each mass. Then $K - P$ goes into the action integral, and there is an Euler equation $\delta A/\delta u_i = 0$ for each mass. In matrix form this is the basic equation of mechanical engineering:

**Undamped oscillation** $$Mu'' + A^{\mathrm{T}}CAu = 0\,.$$

$M$ is the diagonal mass matrix and $A^{\mathrm{T}}CA$ is the positive definite stiffness matrix. The system oscillates around equilibrium and the total energy $K + P$ is constant.

**Example 6** Waves in an elastic bar—*a continuum of masses and springs*.

The action integral in this continuous case has an integral instead of a sum:

**Action** $$A(u) = \int_{t_0}^{t_1}\int_{x=0}^{1}\left[\frac{1}{2}m\left(\frac{du}{dt}\right)^2 - \frac{1}{2}c\left(\frac{du}{dx}\right)^2\right]dx\,dt\,.$$

The Euler-Lagrange rules for $\delta A/\delta u = 0$ cover this case of a double integral:

**Wave equation** $$\frac{\delta A}{\delta u} = -\frac{\partial}{\partial t}\left(m\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left(-c\frac{\partial u}{\partial x}\right) = 0\,.$$

That is the **wave equation** $mu_{tt} = cu_{xx}$. With constant density $m$ and elastic constant $c$, the wave speed is $\sqrt{c/m}$—faster when the bar is stiffer and lighter.

Staying with the calculus of variations, there are two important comments:

**1.** When $u$ is independent of time (no motion), the kinetic energy is $K = 0$ and the least action principle reduces to $\delta P/\delta u = 0$. The dynamic problem goes back to the static problem of $A^{\mathrm{T}}CAu = f$, without a time derivative.

**2.** The dual variable $w$, from all the evidence, should be found in its usual place in Figure 7.7. It is the ***momentum*** $p = mv = m\,du/dt$. We can rewrite $K$:

$$\textbf{The kinetic energy} \quad K = \frac{1}{2}mv^2 \quad \textbf{becomes} \quad \frac{1}{2m}p^2\,.$$

That is the "complementary" kinetic energy, expressed in terms of $p$ instead of $v$. Note that $m$ moves to the denominator, just as $c$ did in the elastic energy $w^2/2c$. Hamilton found the right total energy, using the momentum $p$ and the displacement $u$. Those variables (not the velocity!) are at the primal and dual corners of our framework:

$$\textbf{Hamiltonian} \qquad H = K + P = \frac{1}{2m}p^2 + mgu\,. \tag{21}$$

The Hamiltonian $H(p, u)$ takes us directly to the equations of motion:

$$\textbf{Hamilton's equations} \qquad \frac{\partial H}{\partial p} = \frac{du}{dt} \quad \text{and} \quad \frac{\partial H}{\partial u} = -\frac{dp}{dt}\,. \tag{22}$$

The derivative $dH/dt$ is zero so *the total energy is constant*:

$$\textbf{Chain rule} \qquad \frac{dH}{dt} = \frac{\partial H}{\partial p}\frac{dp}{dt} + \frac{dH}{du}\frac{du}{dt} = u'p' - p'u' = 0\,. \tag{23}$$

This is the essence of classical mechanics. It is tied to Hamilton and not to Newton. For that reason it survived the revolution brought by Einstein. We will see that $H$ has a relativistic form and even a quantum mechanical form. Comparing a falling ball with an oscillating spring, the key difference in $H$ is between $u$ and $u^2$:

$$(\textbf{ball}) \quad H = \frac{1}{2m}p^2 + mgu \qquad\qquad (\textbf{spring}) \quad H = \frac{1}{2m}p^2 + \frac{1}{2}cu^2$$

Hamilton's equations $\partial H/\partial p = u'$ and $\partial H/\partial u = -p'$ yield Newton's Laws:

$$\begin{array}{llll} (\textbf{ball}) & \dfrac{p}{m} = u' \quad \text{and} \quad mg = -p', & \text{or} & mu'' = -mg \\ (\textbf{spring}) & \dfrac{p}{m} = u' \quad \text{and} \quad cu = -p', & \text{or} & mu'' + cu = 0\,. \end{array}$$

The mass passes through equilibrium at top speed (all energy in $K$). The force reverses to slow it down and stop it (all energy in $P$). In the $u - p$ plane (*the phase plane*) the motion stays on the energy surface $H =$ constant, which is the ellipse in Fig. 7.8. Each oscillation of the spring is a trip around the ellipse.

With more springs there are $2n$ axes $u_1, \ldots, u_n, p_1, \ldots, p_n$ and the ellipse becomes an ellipsoid. Hamilton's equations are $\partial H/\partial p_i = du_i/dt$ and $\partial H/\partial u_i = -dp_i/dt$. They lead again to $Mu'' + Ku = 0$, and to the wave equation in the continuous case.

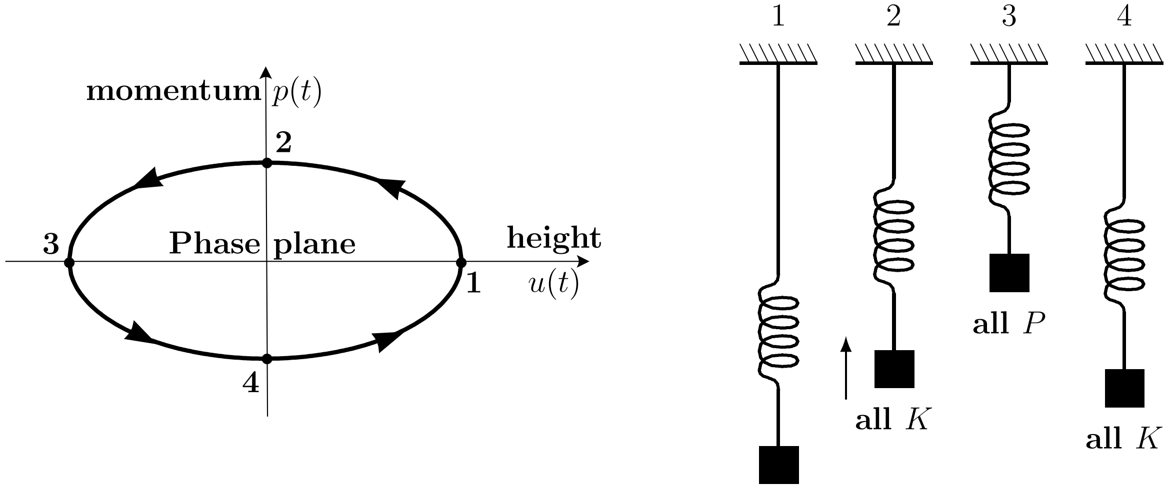



Figure 7.8: **1** tension **2** motion **3** compression **4** motion: Constant energy $H = K + P$.

## Relativity and Quantum Mechanics

These few paragraphs are an attempt, by a total amateur, to correct the action integral by the rules of relativity. Feynman's lectures propose the term $-mc^2\sqrt{1-(v/c)^2}$ in the Lagrangian $K - P$. At $v = 0$ we see Einstein's formula $e = mc^2$ for the energy in a mass $m$ at rest. It becomes part of the potential energy $P$.

As the velocity increases from zero there is also a part corresponding to $K$. For small $x$ the square root of $1-x$ is approximately $1-\frac{1}{2}x$, which linearizes the problem and brings back Newton's $\frac{1}{2}mv^2$—just as linearizing the minimal surface equation brought back Laplace. Relativity mixes together $K$ and $P$!

**Lagrangian** $$F(v) = -mc^2\sqrt{1-(v/c)^2} \approx -mc^2\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right).$$

Now, trusting in duality, we look for the conjugate function $F^*$ in the Hamiltonian. It will be a function of $p$, not $v$. The first step is to find that momentum $p$. Before relativity, $F$ was $\frac{1}{2}mv^2$ and its derivative was $p = mv$. Always $p$ is $\partial F/\partial v$:

**Momentum** $$p = \frac{\partial F}{\partial v} = \frac{mv}{\sqrt{1-(v/c)^2}}. \qquad (24)$$

This becomes infinite as $v$ approaches the speed of light. According to (18), $F^*$ is

**Hamiltonian** $$F^*(p) = \max_v\,[pv - F(v)] = mc^2\sqrt{1+(p/mc)^2}. \qquad (25)$$

This maximum occurs at (24)—which we solved for $v$ and substituted into $pv - F(v)$.

The new Hamiltonian $F^*$ is again Einstein's energy $e = mc^2$ when the system is at rest. As $p$ increases from zero, the next contribution is Newton's $p^2/2m$:

$$F^* \approx mc^2\left(1 + \frac{1}{2}\frac{p^2}{m^2c^2}\right) = \text{ rest energy } + \frac{p^2}{2m}.$$

Newton found the low-order term in the energy $F^*$ that Einstein computed exactly! Perhaps the universe is like a minimal surface in space-time. To Laplace and Newton it looked flat (linearized), but to Einstein it became curved.

It is risky to add anything about quantum mechanics, which works with probabilities. It is a mixture of differential equations (Schrödinger) and matrix equations (Heisenberg). The event at which $\delta A/\delta u = 0$ almost always occurs. Feynman gave each possible trajectory of the system a phase factor $e^{iA/h}$ multiplying its probability amplitude. The small number $h$ (Planck's constant) means that a slight change in the action $A$ completely alters the phase. There are strong canceling effects from nearby paths unless *the phase is stationary.* In other words $\delta A/\delta u = 0$ at the most probable path.

This prediction of "stationary phase" applies equally to light rays and particles. Optics follows the same principles as mechanics, and light travels by the fastest route: ***least action becomes least time***. If Planck's constant could go to zero, the deterministic principles of least action and least time would appear and the path of least action would be not only probable but certain.

## Problem Set 7.2

**1** What are the weak form and the strong form of the linear beam equation—the Euler equation for $P = \int [\frac{1}{2}c(u'')^2 - fu]\,dx$?

**2** Minimizing $P = \int (u')^2 dx$ with $u(0) = a$ and $u(1) = b$ also leads to the straight line through these points. Write down the weak form and the strong form.

**3** Find the Euler equations (strong form) for

$$\text{(a)} \int [(u')^2 + e^u]\,dx \qquad \text{(b)} \int uu'\,dx \qquad \text{(c)} \int x^2(u')^2 dx$$

**4** If $F(u, u')$ is independent of $x$, as in almost all our examples, show from the Euler equation and the chain rule that $H = u'\partial F/\partial u' - F$ is constant. This is dual to the fact that $\partial F/\partial u'$ is constant when $F$ is independent of $u$.

**5** If the speed is $x$ the travel time of a light ray is

$$T = \int_0^1 \frac{1}{x}\sqrt{1 + (u')^2}\,dx \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u(1) = 1\,.$$

(a) From equation $\delta T/\delta u = 0$ what quantity is constant (Snell's law)?

(b) Can you integrate once more to find the optimal path $u(x)$?

**6** With the constraints $u(0) = u(1) = 0$ and $\int u\,dx = A$, show that the minimum value of $P = \int (u')^2 dx$ is $12A^2$. Introduce a multiplier $m$, solve the Euler equation for $u$, and verify that $A = -m/24$. Then the derivative $dP/dA = 24A$ equals the multiplier $-m$ as the sensitivity theory predicts.

**7** For the shortest path constrained by $\int u\,dx = A$, what is unusual about the solution in Fig. 7.5 as $A$ becomes large?

**8** Suppose the constraint is $\int u\,dx \geq A$, with inequality allowed. Why does the solution remain a straight line as $A$ becomes small? Where does the multiplier $m$ remain? This is typical of inequality constraints: either the Euler equation is satisfied or the multiplier is zero.

**9** Suppose the constrained problem is reversed, and we *maximize* the area $P = \int u\,dx$ subject to fixed length $I = \int \sqrt{1+(u')^2}\,dx$, with $u(0) = a$ and $u(1) = b$.

(a) Form the Lagrangian and solve its Euler equation for $u$.

(b) How is the multiplier $M$ related to $m$ in the text?

(c) When do the constraints eliminate all functions $u$?

**10** Find by calculus the shortest broken-line path between $(0,1)$ and $(1,1)$ that goes first to the horizontal axis $y = 0$ and bounces back. Show that the best path treats this axis like a mirror: angle of incidence = angle of reflection.

**11** The principle of maximum entropy selects the probability distribution that maximizes $H = -\int u \log u\,dx$. Introduce Lagrange multipliers for the constraints $\int u\,dx = 1$ and $\int xu\,dx = 1/a$, and find by differentiation an equation for $u$. On the interval $0 < x < \infty$ show that the most likely distribution is $u = ae^{-ax}$.

**12** If the second moment $\int x^2 u\,dx$ is also known show that Gauss wins again: the maximizing $u$ is the exponential of a quadratic. If only $\int u\,dx = 1$ is known, the most likely distribution is $u =$ constant. The *least* information comes when only one outcome is possible, say $u(6) = 1$, since $u \log u$ is then identically zero.

**13** A helix climbs around a cylinder with $x = \cos\theta, y = \sin\theta, z = u(\theta)$:

$$\text{its length is} \quad L = \int \sqrt{dx^2+dy^2+dz^2} = \int \sqrt{1+(u')^2}\,d\theta\,.$$

Show that $u' =$ constant satisfies Euler's equation. The helix is regular.

**14** Multiply the nonlinear equation $-u'' + \sin u = 0$ by $v$ and integrate the first term by parts to find the weak form. What integral $P$ is minimized by $u$?

**15** Find the Euler equations (strong form) for

(a) $P(u) = \dfrac{1}{2}\displaystyle\iint \left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2\right] dx\,dy$

(b) $P(u) = \dfrac{1}{2}\displaystyle\iint (yu_x^2 + u_y^2)\,dx\,dy$ (c) $E(u) = \displaystyle\int u\sqrt{1+(u')^2}\,dx$

(d) $P(u) = \dfrac{1}{2}\displaystyle\iint (u_x^2 + u_y^2)\,dx\,dy$ with $\displaystyle\iint u^2\,dx\,dy = 1$.

**16** Show that the Euler equations for these integrals are the same:

$$\iint \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}\, dx\, dy \quad \text{and} \quad \iint \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 dx\, dy$$

Presumably the two integrals are equal if the boundary conditions are zero.

**17** Sketch the graph of $p^2/2m + mgu =$ constant in the $u - p$ plane. Is it an ellipse, parabola, or hyperbola? Mark the point where the ball reaches maximum height and begins to fall.

**18** Draw a second spring and mass hanging from the first. If the masses are $m_1, m_2$ and the spring constants are $c_1, c_2$, the energy is

$$H = K + P = \frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2 + \frac{1}{2}c_1 u_1^2 + \frac{1}{2}c_2(u_2 - u_1)^2 .$$

Find the four Hamilton's equations $\partial H/\partial p_i = du_i/dt, \partial H/\partial u_i = -dp_i/dt$, and the matrix equation $Mu'' + Ku = 0$.

**19** The Hamiltonian for a pendulum (with $u = \theta$) is $H = p^2/2m + mgl(1 - \cos u)$. Write out Hamilton's equations (22) and eliminate $p$ to find the equation of a pendulum.

**20** Verify that the energy $\frac{1}{2}e^{\mathrm{T}}Ce$ and the complementary energy $\frac{1}{2}w^{\mathrm{T}}C^{-1}w$ are conjugate. As in equation (18), this means that $\frac{1}{2}w^{\mathrm{T}}C^{-1}w = \max[e^{\mathrm{T}}w - \frac{1}{2}e^{\mathrm{T}}Ce]$.