

5.5 Difference Matrices and Eigenvalues

This brief section collects together useful notes on finite difference matrices, and also finite element matrices. Certainly those special matrices K, T, B, C from the start of the book are the building blocks for approximations to u_{xx} and u_{yy}. Second derivatives and fourth derivatives lead to symmetric matrices. *First derivatives are antisymmetric*. They present more difficulties.

A symmetric matrix has orthogonal eigenvectors. For those special matrices, the eigenvectors are discrete samples of sines and cosines and e^{ikx}. The eigenvalues are real, and they often involve $e^{ik\Delta x}-2+e^{-ik\Delta x}$. That is the discrete factor $2\cos k\Delta x-2$. Divided by $(\Delta x)^2$, it is close for small k to the factor $-k^2$ from the second derivative of e^{ikx}. *The von Neumann approach using e^{ikx} matches the eigenvectors of these matrices*, and the growth factors G match the eigenvalues.

For a one-sided (upwind) difference, the matrix eigenvalues are not always reliable. For a centered difference they do follow von Neumann. Compare

$$\boldsymbol{\Delta}^{+} = \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix} \qquad \boldsymbol{\Delta}^{\mathbf{0}} = \frac{1}{2}\begin{bmatrix} 0 & 1 & & \\ -1 & 0 & -1 & \\ & -1 & 0 & -1 \\ & & -1 & 0 \end{bmatrix}$$

The eigenvalues of the triangular upwind matrix Δ^+ are all -1 (*useless*). The eigenvalues of the antisymmetric Δ^0 are guaranteed to be imaginary like the factor ik from the derivative of e^{ikx}. The eigenvalues $\lambda = -1$ for Δ^+ do not make upwind differences useless. They only mean that the von Neumann test, which produces $e^{ik\Delta x}-1$, is better than relying on eigenvalues.

As it stands, Δ^+ is exactly in "*Jordan form*." The matrix has only one line of eigenvectors, not n. It is an extreme example of a nondiagonalizable (and somehow degenerate) matrix. If the diagonals of -1's and 1's are extended to infinity, then Fourier and von Neumann produce vectors with components $e^{ikj\Delta x}$ and with eigenvalues $e^{ik\Delta x}-1$. In summary: For normal matrices, eigenvalues are a reliable guide. For other constant-diagonal matrices, better to rely on von Neumann.

Briefly, the discrete growth factors G are exactly the eigenvalues when the matrices are called "*normal*" and the test is $AA^{\mathrm{T}} = A^{\mathrm{T}}A$ (for complex matrices take the conjugate transpose A^*). This test is passed by all symmetric and antisymmetric and orthogonal matrices.

Options for First Differences

Upwind elements
Streamline diffusion
DG
Boundary conditions
Convection-diffusion