18.085 SUMMER 2014 - QUIZ 3 - AUGUST 15, 2014

YOUR NAME: ______________________________

YOUR SCORE: ____________ **/ 100 +** ____________ **BONUS**

This exam has 5 questions!

Question 1 (5+8+3+3+6=25pts) Real and complex Fourier series.

Let $f(x) = |x|$ be defined on $-\pi \leq x < \pi$, and let $g(x)$ be its 2π-periodic extension.

(a) Sketch the function $g(x)$. Label its maximum value(s), and the location(s) of the maximum value(s), on your graph.

(b) Compute the real Fourier coefficients of $g(x)$. You may use the symmetry of the function to reduce your workload, and use integration by parts.

(c) What are the complex Fourier coefficients of $g(x)$? You may use your answer from part (b) to reduce your workload.

(d) How fast do the Fourier coefficients of $g(x)$ decay as $n \to \infty$? Explain why you could have predicted this without doing parts (b) or (c).

(e) Use Parseval's theorem to deduce the following equality:

$$\sum_{n \geq 1,\text{ odd}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$$

Question 2 (3+3+4+6=16 pts) The Discrete Fourier Transform Part 1.

Let $w = e^{i2\pi/N}$.

(a) The entries of the Fourier matrix F are $F_{jk} =$ ________________.

(b) The entries of the inverse of the Fourier matrix F^{-1} are $F^{-1}_{jk} =$ ________________.

Let $y(x)$ be the following function (assumed to be 2π-periodic):

$$y(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi. \end{cases}$$

(c) Sample $y(x)$ at the $N = 3$ Fourier sample points to get the sample vector $\vec{y} = (y(0), \cdots)$:

$$\vec{y} = (\underline{\hspace{6cm}})^T.$$

(d) Find the discrete Fourier coefficients $\vec{c}$ of $\vec{y}$ by solving the equation $\vec{y} = F\vec{c}$, again for $N = 3$. The answer uses numbers, no w's and no square roots! You might want to recall that columns and rows of F sum to 0, except for the 0th row and column.

Question 3 (6+6+4=16 pts) The Discrete Fourier Transform Part 2.

You are given the vector $\vec{c} = \frac{1}{3}(2, -1, -1)^T$ (different from part 1).

(a) Take the cyclic convolution of the vector $\vec{c}$ with itself: $\vec{c} \circledast \vec{c}$. You can use the direct method you like ("long multiplication" or using a circulant matrix), but DO NOT use the convolution rule.

(b) Multiply the Fourier matrix F (for $N = 3$) with the convolution of (a). That is, calculate $F(\vec{c} \circledast \vec{c})$. The answer uses numbers, not w's.

(c) Let $\vec{y} = F\vec{c} = (0, 1, 1)^T$, same $\vec{c}$ as above. Is the following equation true? YES or NO. No need to justify, just circle your answer.

$$\begin{pmatrix} y_1 y_1 \\ y_2 y_2 \\ y_3 y_3 \end{pmatrix} = F((F^{-1}\vec{y}) \circledast (F^{-1}\vec{y})).$$

Question 4 ($5+3+6+6+3=23$ pts + 5 BONUS) Fourier Integral Transform.

This question uses the Fourier Integral Transform. Let $f(x)$ be the following function, defined over the whole real line $\mathbb{R}$ (NOT 2π-periodic). It looks like a hat.

$$f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Draw functions $f(x)$, $g(x) = df/dx$ and $h(x) = d^2f/dx^2$ on 3 separate graphs. Make sure to do this right, it'll be useful in (b), (d) and maybe even (f).

(b) The solution $u(x)$ to equation (1) (no calculation needed) is $u(x)=$ ______________ .

$$-\frac{d^2u}{dx^2} = \delta(x) - 2\delta(x-\pi) + \delta(x-2\pi) \tag{1}$$

(c) Take the Fourier Integral Transform (FIT) of equation (1) to find an expression for $\hat{u}(k)$, the FIT of $u(x)$. The answer $\hat{u}(k)$ involves only numbers and k's, which means you'll need to calculate the FIT of those delta functions.

Question 4 continued. $(5+3+6+6+3=23$ **pts + 5 BONUS)**

(d) Find the FIT $\hat{g}(k)$ of $g(x) = df/dx$ using the formula $\hat{g}(k) = \int_{-\infty}^{\infty} g(x)e^{-ikx}dx$.

(e) From your answer in (d), find the FIT $\hat{f}(k)$ of $f(x)$.

You can check your answer using (b) and (c) if you want! Only partial credit awarded if you use (b) and (c) but not (d) to find $\hat{f}(k)$.

(f) BONUS QUESTION. OPTIONAL. Find the FIT $\hat{h}(k)$ of $h(x)$. You may use either the work you did in (c) or in (d) (or use both to check your answer if you'd like).

Question 5 ($6+6+5+3=20$ pts) Filtering.

Consider the following filter, which takes an infinite vector input $\vec{x}$ and outputs the infinite vector $\vec{y}$ as such:

$$y_k = x_k - \frac{1}{2}x_{k-1} - \frac{1}{2}x_{k-2}, \qquad -\infty \leq k \leq \infty, \qquad k \text{ integer.}$$

(a) Let $\vec{h}$ be the infinite vector such that the convolution $\vec{y} = \vec{h} * \vec{x}$ holds. It has only 3 non-zero components. Write down these 3 components (value + which k).

(b) Let the input have a frequency of ω: $x_k = e^{ik\omega}$. Calculate the output y_k. Also, write the output as $y_k = x_k H(\omega)$. What is this function $H(\omega)$?

(c) Fill the blanks: $H(0) =$ ______________ and $H(\pi) =$ ______________ . (If you couldn't do (b), you might still be able to try this one out.)

(d) Is this filter LOWPASS or HIGHPASS (circle one)? No justification needed.