18.085 SUMMER 2014 - EXAM 1 - SOLUTIONS

(1) (7 + 8 = 15 points.) In this question, we set up the problem to fit a parabola $b = C + Dt + Et^2$ through these four (t, b) points, minimizing the sum of the squares of the error:

$$(0,1),(2,4),(1,3),(-1,0).$$

(a) Give the matrix A and the vector $\vec{b}$ for the least squares formulation of this problem.

Solution.

The first column of A corresponds to the constant C, so it contains all ones. The second column corresponds to the line of slope D, so it contains the t's. The third column corresponds to the parabola with coefficient E, so it contains the t^2's.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

The observations in $\vec{b}$ correspond to the b's:

$$\vec{b} = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 0 \end{pmatrix}$$

(b) Multiply A^T with A to get A^TA, and A^T with $\vec{b}$ to get $A^T\vec{b}$, and give your results.

Solution.

We know A^TA will be symmetric, so we do not need to repeat calculations.

$$A^TA = \begin{pmatrix} (1+1+1+1) & (0+2+1-1) & (0+4+1+1) \\ 2 & (0+4+1+1) & (0+8+1-1) \\ 6 & 8 & (0+16+1+1) \end{pmatrix} = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{pmatrix}.$$

$$A^T\vec{b} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 4 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \\ 19 \end{pmatrix}.$$

(2) (15 points.)

Find the LU decomposition of the matrix M below using elimination. **Show your steps!**

$$M = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{pmatrix} \xrightarrow[R_3 \to R_3 - (6/4)R_1]{R_2 \to R_2 - (2/4)R_1} \begin{pmatrix} 4 & 2 & 6 \\ 0 & 5 & 5 \\ 0 & 5 & 9 \end{pmatrix}, \qquad l_{21} = 1/2,\ l_{31} = 3/2.$$

$$\xrightarrow[R_3 \to R_3 - (5/5)R_2]{} \begin{pmatrix} 4 & 2 & 6 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{pmatrix}, \qquad l_{31} = 1.$$

So we obtain our matrices L and U:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 4 & 2 & 6 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{pmatrix}.$$

(3) (15 points.)

You are given the LU decomposition of a matrix M: $M = LU$ where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 4 & 2 & 6 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{pmatrix}.$$

Use the LU decomposition of M to solve the system of equations $M\vec{x} = \vec{y}$, where $\vec{y} = (8\ 11\ 19)^T$. Recall how you can do this in two easy steps, first using L, then using U.

Solution.

First, we solve for $\vec{x}$ in $L\vec{x} = \vec{y}$, substituting starting from the top. The top row tells us $x_1 = 8$. Using that, the second row tells us that $x_2 = (11 - 8 \times 1/2)/1 = 7$. Finally, the third row tells us $x_3 = (19 - 3/2 \times 8 - 1 \times 7)/1 = 0$.

$$L\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \\ 19 \end{pmatrix} = \vec{b}.$$

Then, we solve for $\vec{u}$ in $U\vec{u} = \vec{x}$, substituting starting from the bottom. The last row tells us $u_3 = 0/4 = 0$. The second row says $u_2 = (7 - 5 \times 0)/5 = 7/5$. The first row says $u_1 = (8 - 7/5 \times 2 - 6 \times 0)/4 = (40/5 - 14/5)/4 = 26/5/4 = 13/10$.

$$U\vec{u} = \begin{pmatrix} 4 & 2 & 6 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 13/10 \\ 7/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \\ 0 \end{pmatrix} = \vec{x}.$$

(4) $(5+10+5+10=30$ points.)

We wish to solve the following convection-diffusion equation using finite differences:

$$-D\frac{d^2}{dx^2}u(x)+\frac{d}{dx}u(x)=\delta(x-1/2), \qquad u(0)=0, \qquad u(1)=0$$

where D is a positive scalar, the diffusion coefficient. That is, we will obtain a system

$$\frac{D}{h^2}K\vec{u}+A\vec{u}=\vec{\delta}, \qquad h=1/(n+1), \qquad K \text{ is } n\times n. \tag{1}$$

(a) What should be the dimensions of matrix A? **Solution: A is n by n, like K.**

(b) Use a forward difference[1] for the du/dx term. What will be the matrix A multiplying the vector $\vec{u}$ in (1), for a general n (assuming n is odd)? Be careful at the boundaries, and so make sure you clearly show what the first and last rows and columns will be!

Solution.

Because we have fixed-fixed boundary conditions, the first row and column correspond to $x=h$ and the last row and column correspond to $x=nh$. At $x=h$, a forward difference means

$$u'(h)\approx\frac{u(2h)-u(h)}{h}$$

so that the first row of matrix A should start with $-1/h$ (for $u_1\approx u(h)$), then $1/h$ (for $u_2\approx u(2h)$), then all zeros. As for the last column, a forward difference means

$$u'(nh)\approx\frac{u((n+1)h)-u(nh)}{h}, \quad u((n+1)h)=u(1)=0$$

so that the last row of matrix A should end with $-1/h$ (for $u_n\approx u(nh)$), and be all zeros otherwise. From this we can see the following pattern:

$$A=\frac{1}{h}\begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & & \\ 0 & -1 & 1 & 0 & \cdots & & \\ \vdots & & \ddots & \ddots & & & \\ & & & & & & \\ & & & & & -1 & 1 \\ & & & & & 0 & -1 \end{pmatrix}.$$

[1] It turns out that for a convective term such as u', a forward difference is better than a centered one, but we have no time to explain this here.

(c) Use the typical vector approximation to the delta function. Write down clearly what $\vec{\delta}$ is when approximating the term $\delta(x-1/2)$, for a general odd n.

Solution.

The delta vector consists of all zeros, except one entry corresponding to, in this case, $x=1/2$. This entry is $1/h$. In the case of $x=1/2$, the correct entry is entry $(n+1)/2$, in the very middle of the vector: $\vec{\delta}=(0,\cdots,0,1/h,0,\cdots)^T$.

(d) This question is now about the matrix DK/h^2. Is this matrix positive definite? Show your steps, and in particular, **say which test you will use, then use that test.** OK to use any knowledge you have of K without proving that.

Solution.

You did not have to use both tests, but we present both here for completeness.

(a) Test 4: we know that the energy $\vec{x}^T K\vec{x}>0$ because K is positive definite. Since the scalar D/h^2 is positive, then the energy of the matrix DK/h^2 will be the same as that of K except multiplied by the positive scalar D/h^2. This means that the matrix DK/h^2 is also positive definite.

(b) Test 3: since K is positive definite, then it has positive eigenvalues. But the eigenvalues of matrix DK/h^2 are the same as those of K, except multiplied by the positive scalar D/h^2. This means the matrix DK/h^2 is positive definite as well. To see the eigenvalues of matrix DK/h^2, consider first an eigenvalue λ and an eigenvector $\vec{y}$ of K: $K\vec{y}=\lambda\vec{y}$. Then, $DK/h^2\ \vec{y}=(D/h^2)K\vec{y}=(D/h^2)\lambda\vec{y}=(D\lambda/h^2)\vec{y}$, so that $(D\lambda/h^2)$ is an eigenvalue of matrix DK/h^2 with eigenvector $\vec{y}$.

(5) ($15 + 10 = 25$ points.)

We have seen in class multiple ways of solving the equation $u''(t) + u(t) = 0$. Here's (I think!) the last one. First, we rewrite this as a system with only first derivatives:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}. \tag{2}$$

(a) Find the eigenvalues λ_1, λ_2 and eigenvectors y_1, y_2 (they might be complex) of the matrix A above.

Solution.

First, we find the eigenvalues, by putting the determinant of $A - \lambda I$ equal to 0:

$$det(A - \lambda I) = \left|\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}\right| = \lambda^2 + 1 = 0 \Rightarrow \lambda_1 = i, \ \lambda_2 = -i$$

where $i = \sqrt{-1}$.

Next, we find the eigenvectors, by looking for vectors $\vec{y}$ such that $(A - \lambda I)\vec{y} = \vec{0}$:

$$(A - \lambda_1 I)\vec{y_1} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} y_1^{(1)} \\ y_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we can see that $\vec{y_1} = (i, \ -1)^T$ works, and

$$(A - \lambda_2 I)\vec{y_1} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} y_2^{(1)} \\ y_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we can see that $\vec{y_2} = (i, \ 1)^T$ works.

(b) Find the two solutions $(u_1, \ v_1)^T$ and $(u_2, \ v_2)^T$ of the system (2), using eigenvectors and eigenvalues. (You do not need to have found the actual eigenvectors and eigenvalues in (a) to answer this.)

We have done this before:

$$(u_1(t), \ v_1(t))^T = e^{\lambda_1 t}\vec{y_1}, \qquad (u_2(t), \ v_2(t))^T = e^{\lambda_2 t}\vec{y_2}$$