18.085 SUMMER 2014 - EXAM 1 - JUNE 30, 2014

YOUR NAME: __________________________________

YOUR SCORE: _______________ **/ 100**

Don't forget to show your steps!

(1) ($7 + 8 = 15$ points.) In this question, we set up the problem to fit a parabola $b = C + Dt + Et^2$ through these four (t, b) points, minimizing the sum of the squares of the error:

$$(0,1),(2,4),(1,3),(-1,0).$$

(a) Give the matrix A and the vector $\vec{b}$ for the least squares formulation of this problem.

(b) Multiply A^T with A to get A^TA, and A^T with $\vec{b}$ to get $A^T\vec{b}$, and give your results. Do not solve for the best fit. **Show your steps!**

(2) (15 points.)

Find the LU decomposition of the matrix M below using elimination. **Show your steps!**

$$M = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{pmatrix}.$$

(3) (15 points.)

You are given the LU decomposition of a matrix M: $M = LU$ where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 4 & 2 & 6 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{pmatrix}.$$

Use the LU decomposition of M to solve the system of equations $M\vec{x} = \vec{y}$, where $\vec{y} = (8\ 11\ 19)^T$. Recall how you can do this in two easy steps, first using L, then using U. **Show your steps!**

(4) ($5 + 10 + 5 + 10 = 30$ points.)

We wish to solve the following convection-diffusion equation using finite differences:

$$-D\frac{d^2}{dx^2}u(x) + \frac{d}{dx}u(x) = \delta(x - 1/2), \qquad u(0) = 0, \qquad u(1) = 0$$

where D is a positive scalar, the diffusion coefficient. That is, we will obtain a system

$$\frac{D}{h^2}K\vec{u} + A\vec{u} = \vec{\delta}, \qquad h = 1/(n+1), \qquad K \text{ is } n \times n. \tag{1}$$

(a) **What should be the dimensions of matrix A?**

(b) Use a forward difference[1] for the du/dx term. What will be the matrix A multiplying the vector $\vec{u}$ in (1), for a general n (assuming n is odd)? Be careful at the boundaries, and make sure in particular that you clearly show what the first and last rows and columns will be!

(c) Use the typical vector approximation to the delta function. Write down clearly what $\vec{\delta}$ is when approximating the term $\delta(x - 1/2)$, for a general odd n. **Be as precise as you can.**

[1]It turns out that for a convective term such as u', a forward difference is better than a centered one, but we have no time to explain this here.

(d) This question is now about the matrix DK/h^2. Is this matrix positive definite? Show your steps, and in particular, **say which test you will use, then use that test.**

(5) $(15 + 10 = 25$ points.)

We have seen in class multiple ways of solving the equation $u''(t) + u(t) = 0$. Here's (I think!) the last one. First, we rewrite this as a system with only first derivatives:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}. \tag{2}$$

(a) Find the eigenvalues λ_1, λ_2 and eigenvectors y_1, y_2 (they might be complex) of the matrix A above.

(b) Find the two solutions $(u_1,\ v_1)^T$ and $(u_2,\ v_2)^T$ of the system (2), using eigenvectors and eigenvalues. (You do not need to have found the actual eigenvectors and eigenvalues in (a) to answer this.)