18.085, QUIZ 3 2008 SOLUTIONS

Question 2.
(a) We have 8 equations for the 8 c's:

$$\begin{aligned} p(1) = 1 &\Rightarrow \quad c_0 + c_1 + c_2 + \cdots + c_7 = 1 \\ p(w) = 0 &\Rightarrow \quad c_0 + c_1 w + c_2 w^2 \cdots + c_7 w^7 = 0 \\ p(w^2) = 1 &\Rightarrow \quad c_0 + c_1(w^2) + c_2(w^2 2)^2 \cdots + c_7 (w^2)^7 = 0 \\ \cdots & \end{aligned}$$

so that the system is $A\vec{c} = \vec{b}$ where A is in fact the Fourier matrix! That is, $A_{jk} = w^{jk}$. The vector $\vec{c}$ of course is the vector of the c_j's, the polynomial's coefficients. The vector $\vec{b}$ is $\vec{b} = (1, 0, 1, 0, 1, 0, 1, 0)^T$.

(b) We can solve for $\vec{c} = A^{-1}\vec{b}$ since $A^{-1} = F^{-1}$, with $A^{-1}_{jk} = \frac{1}{N}\bar{w}^{jk}$. We can see that we will only need the even columns (starting at the 0th column) of the inverse. When $k = 2l$ is even, then $\bar{w}^{jk} = w^{-j2l} = (w^2)^{-jl}$ where $w^2 = (e^{i2\pi/8})^2 = e^{i2\pi/4} = i$. We will need this in the next question.

(c) We find $\vec{c}$:

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \\ \\ \\ \\ \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w^{-1} & -i & w^{-3} & (-i)^2 & w^{-5} & (-i)^3 & w^{-7} \\ 1 & w^{-2} & (-i)^2 & w^{-6} & \cdots & & & \\ \vdots & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 + 1 + 1 + 1 \\ 1 + (-i) + (-i)^2 + (-i)^3 \\ 1 + (-i)^2 + (-i)^4 + (-i)^6 \\ \vdots \end{pmatrix}.$$

Here you can either compute these sums and see that most of them are 0, or you can reason by remembering that the row and columns of the Fourier matrix and its inverse sum to 0, except for the 0th row. In particular, the rows of the inverse Fourier matrix with $N = 4$ sum to 0, and this is what we get for all c_j's except c_0 and c_4. For c_0 as we can see the sum is 1/2. As for c_4, the sum is $c_4 = (1 + (-i)^4 + (-i)^8 + (-i)^{12})/8 = (1 + 1 + 1 + 1)/8 = 1/2$ as well. Again, $(-i)^4 = 1$ so $(-i)^8 = 1$ as well, etc. So finally, $\vec{c} = (1/2, 0, 0, 0, 1/2, 0, 0, 0)^T$.

This means

$$p(z) = \frac{1}{2}\left(1 + z^4\right)$$

and we can check that $p(w^j) = (1 + w^{4j})/2$, where as we know $w^{4j} = (w^4)^j = (e^{i2\pi/2})^j = (-1)^j$. This means $p(w^j) = (1 + (-1)^j)/2$, as required.