18.085, PROBLEM SET 8 SOLUTIONS

Question 1. (15 pts.) Question 4 of pset7, continued.
Let $f(x)$ be the 2π-periodic extension of the function $g(x) = e^{-x}$ defined on the interval $-\pi \le x \le \pi$. See the solutions to pset7 online for the answers to a) (sketch of f) and b) the complex Fourier series for f.

c) Find the Fourier series of df/dx from the Fourier series of f: we know the complex Fourier series of f is

$$f(x) = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+in} e^{inx}$$

so, taking the x derivative term by term, we obtain the complex Fourier series of f':

$$f'(x) = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+in} (in) e^{inx}$$

d) Now find the Fourier series of the sum $f' + f$:

$$f + f' = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+in} (1+in) e^{inx} = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n e^{inx}.$$

Note that $g' + g = 0$, but the series for $f' + f$ isn't 0. This is because f, as a periodic extension of g, is not continuous: it has a jump at $-\pi$, π, etc. Every time there is a jump in f, we get a delta function when taking the derivative. Since the jump in f is $e^{-(-\pi)} - e^{-\pi} = 2\sinh \pi$ (top value, at $x = -\pi$, minus the bottom value at $x = \pi$), we expect to have a coefficient of $2 \sinh \pi$ in front of the complex series for the periodic delta function centered at $x = \pi$ (or $x = -\pi$, etc). We found the complex series of the delta function at $x = 0$ in class, and we also mentioned the shift formula. Or, you can simply recalculate the Fourier coefficients of the delta function $\delta(x - \pi)$. Either way, you obtain :

$$\delta(x - \pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-in\pi} e^{inx} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n e^{inx}$$

as the complex Fourier series of the shifted delta function (the above equality comes from the fact that $e^{-i\pi} = -1$ so $e^{-i\pi n} = (-1)^n$). Multiplying this series by the factor of $2 \sinh \pi$ mentioned above, we recover the series for $f + f'$, as expected.

Question 2. (25 pts.) Decay and derivatives.

Let $f(x) = x(\pi - x)$ over $0 \le x \le \pi$, and let g be its odd extension, that is, $g(x) = f(x)$ for $x \in [0, \pi]$ and $g(x) = -f(-x)$ for $x \in [-\pi, 0]$. We shall investigate g, and its first two derivatives, knowing that the Fourier series of g, an odd function, is:

$$g(x) = \frac{8}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots\right).$$

a) In one plot, over the interval $x \in [\pi/3, 2\pi/3]$ only, we show the function g and its first three partial sums.

Using this code we obtain figure 1. We see g in blue, and we see that the first partial sum, $g1$, is not a very good approximation of g, but that $g2$ does better, and $g3$ is pretty close to g. That is, the approximation of g gets better as we take more and more terms in its Fourier series as expected.

```
x=0:.001:pi;
g=x.*(pi-x);
g1=8/pi*(sin(x));
g2=8/pi*(sin(x)+sin(3*x)/27);
g3=8/pi*(sin(x)+sin(3*x)/27+sin(5*x)/125);
plot(x,g,'-b',x,g1,'-g',x,g2,'-m',x,g3,'-r');
legend('g','g1','g2','g3')
axis([pi/3 2*pi/3 2.1 2.6])
xlabel('x');ylabel('g(x)')
```

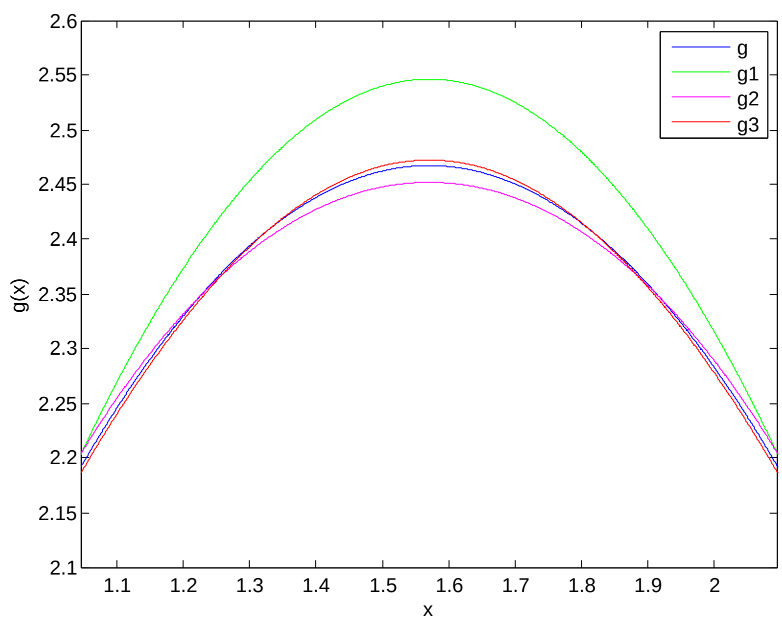

FIGURE 1. The function g and its first 3 partial Fourier sums.

b) The first derivative in x of g is $\pi - 2x$ over $[0, \pi]$ and since g is $x(\pi + x)$ over $[0, \pi]$ its first derivative there is $\pi + 2x$. This is still a continuous function! The first derivative in x of its Fourier series is found by taking the x derivative of each term:

$$g'(x) = \frac{8}{\pi}\left(\frac{\cos x}{1} + \frac{3\cos 3x}{27} + \frac{5\cos 5x}{125} + \cdots\right).$$

c) In one plot, over the interval $x \in [\pi/3, 2\pi/3]$ only, we show the function g' and its first three partial sums. Using this code we obtain figure 2. We see g' in blue, and we see that the first partial sum, $gp1$, is not a very good approximation of g', but that $gp2$ does better, and $gp3$ is pretty close to g'. That is, the approximation of g' gets better as we take more and more terms in its Fourier series as expected.

```
x=0:.001:pi;
gp=pi-2*x;
gp1=8/pi*(cos(x));
gp2=8/pi*(cos(x)+cos(3*x)/9);
gp3=8/pi*(cos(x)+cos(3*x)/9+cos(5*x)/25);
plot(x,gp,'-b',x,gp1,'-g',x,gp2,'-m',x,gp3,'-r');
legend('gp','gp1','gp2','gp3')
axis([pi/3 2*pi/3 -1.5 1.5])
xlabel('x');ylabel('gp(x)')
```

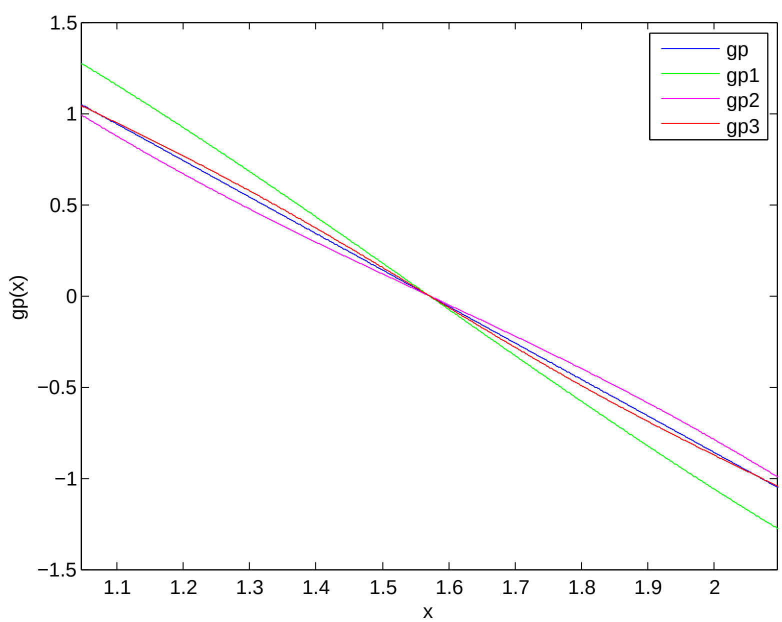

FIGURE 2. The function g' and its first 3 partial Fourier sums.

d) The second derivative in x of g is -2 over $[0,\pi]$ and 2 over $[-\pi,0]$. This is discontinuous! The second derivative in x of its Fourier series is

$$g''(x)=\frac{8}{\pi}\left(-\frac{\sin x}{1}-3^2\frac{\sin 3x}{27}-5^2\frac{\sin 5x}{125}+\cdots\right)=-\frac{8}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right).$$

e) In one plot, over the interval $x\in[\pi/3,2\pi/3]$ only, we show the function g'' and its first three partial sums. Using this code we obtain figure 3. We see g'' in blue, and we see that the first partial sum, $gpp1$, is not a very good approximation of g', but that $gpp2$ does better, and $gpp3$ still better. That is, the approximation of g' gets better as we take more and more terms in its Fourier series as expected. However, because the function is discontinuous, we expect to see Gibb's phenomenon. This is why we see the approximations $gpp1$, $gpp2$ and $gpp3$ oscillate around the correct value of -2.

```
x=0:.001:pi;
gpp=-2*ones(size(x));
gpp1=-8/pi*(sin(x));
gpp2=-8/pi*(sin(x)+sin(3*x)/3);
gpp3=-8/pi*(sin(x)+sin(3*x)/3+sin(5*x)/5);
plot(x,gpp,'-b',x,gpp1,'-g',x,gpp2,'-m',x,gpp3,'-r');
legend('gpp','gpp1','gpp2','gpp3')
axis([pi/3 2*pi/3 -2.6 -1.6])
xlabel('x');ylabel('gpp(x)')
```

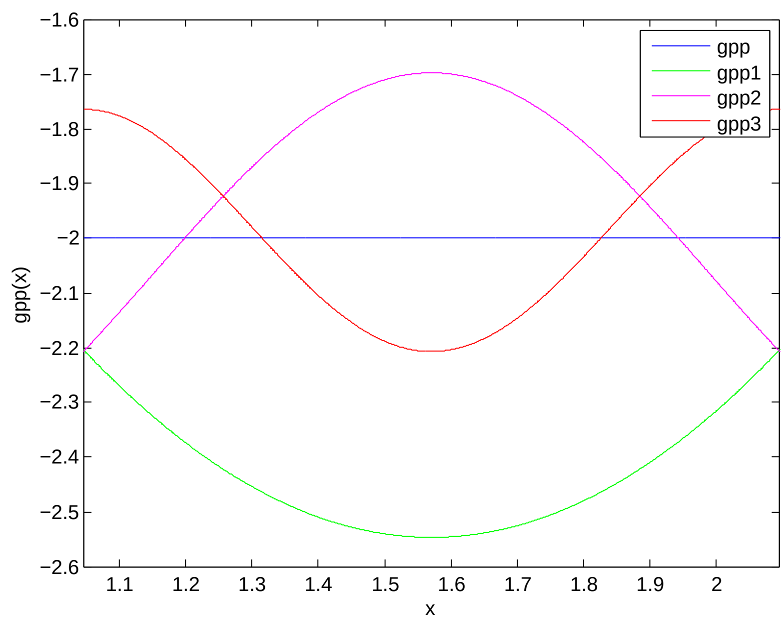

FIGURE 3. The function g'' and its first 3 partial Fourier sums.

f) We can read off the decay rate of the Fourier coefficients from the series we obtained previously. For example, looking at

$$g(x) = \frac{8}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots\right),$$

we see the coefficients are

$$\frac{8}{\pi}\frac{1}{1^3}, \ \frac{8}{\pi}\frac{1}{3^3}, \ \frac{8}{\pi}\frac{1}{5^3}, \ldots$$

which is a decay of $1/n^3$ for g. Similarly, we have a decay of $1/n^2$ for dg/dx and $1/n$ for d^2g/dx^2. This makes sense! Since d^2g/dx^2 is discontinuous at one point, we expect its coefficients to decay like $1/n$. Then, clearly, dg/dx is continuous but has discontinuous first derivative (as we just saw, d^2g/dx^2 is discontinuous) so its coefficients decay like $1/n^2$. And as for g, it has a continuous first derivative, but discontinuous second derivative, so we expect it to have coefficients that decay like $1/n^3$.

g) The function in (d) is just -2 times the square wave function. Hence its Fourier series should be -2 times that of the square wave:

$$-2 \times \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right].$$

This is exactly the fourier series of d^2g/dx^2 over $[-\pi, \pi]$ we found in (d).

Question 3. (30 pts.) The Heat Equation.

We consider the temperature $u = u(x,t)$ in a rod, over the interval $x \in [0,\pi]$ and for time $t > 0$. This temperature must satisfy the heat equation:

$$u_t = u_{xx}, \ x \in [0,\pi], \ t > 0. \tag{0.1}$$

At time $t = 0$, we are given that the temperature is uniform and of value 3 (initial condition):

$$u(x,0) = 3, \ x \in [0,\pi]. \tag{0.2}$$

The rod is coated so that heat can only escape at the ends. At time $t = 0$, the rod is moved into a freezer (zero temperature). This will cause the ends of the rod to be at temperature 0 for all positive times (boundary conditions):

$$u(0,t) = u(\pi,t) = 0, \ t > 0. \tag{0.3}$$

In this question, we solve the heat equation using Fourier series. The boundary conditions (0.3) of 0 imply we should use a sine series (the sines will be automatically 0 where they should). This means we need the initial condition to be odd (so it has a sines series).

a) This is the sketch the odd extension of the function $f(x) = 3$ from the interval $x \in [0,\pi]$ to the interval to $x \in [-\pi,\pi]$. This is 3 times the square wave!

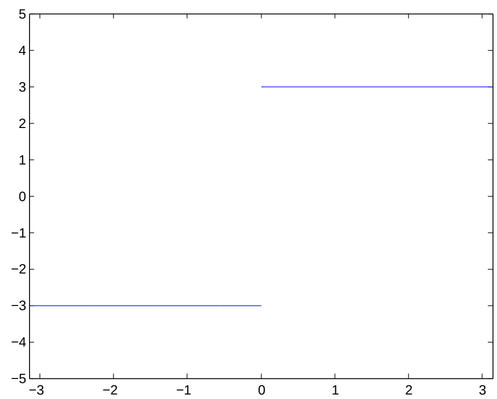

FIGURE 4. The odd extension of f.

b) The sine Fourier series of the odd extension in a) issimply 3 times the square wave:

$$\frac{12}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right].$$

Now we write the solution u (unknown at this point) in a sine series as well, but with coefficients (to be found) that vary in time:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx. \tag{0.4}$$

Notice how now the solution u in the interval $x \in [-\pi, 0]$ is the odd extension of u in the interval $x \in [0, \pi]$. Just like we did with the initial condition.

c) Plug in the solution (0.4) into the heat equation (0.1), to obtain a differential equation that the b_n's must satisfy:

$$u_t(x,t) = \sum_{n=1}^{\infty} b_n'(t) \sin nx = u_{xx}(x,t) = \sum_{n=1}^{\infty} -n^2 b_n(t) \sin nx.$$

Matching the terms, we get that $b_n'(t) = -n^2 b_n(t)$ for $n = 1$ to ∞. This is the required differential equation that the b_n's need to satisfy.

d) Solve the differential equation you obtained in (c), assuming initial conditions $b_n(0)$ for each n: the solution is $b_n(t) = b_n(0)e^{-n^2 t}$. (Remember from a few weeks ago how the solution to $v' = av$ was $v = v(0)e^{at}$.)

All that remains is to find the actual initial conditions $b_n(0)$! We know u must satisfy the initial condition (0.2), and we have sine Fourier series for each.

e) Match the terms in the Fourier series of u and the initial condition to obtain the value of the $b_n(0)$, for each n: we have u which is now

$$u(x,t) = \sum_{n=1}^{\infty} b_n(0)e^{-n^2 t} \sin nx,$$

or (when evaluated at $t = 0$):

$$u(x,0) = \sum_{n=1}^{\infty} b_n(0) \sin nx,$$

and f which is

$$\frac{12}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right].$$

Matching terms, we get that $b_n(0) = 12/(n\pi)$ when n is odd, and $b_n(0) = 0$ when n is even. Hence the solution to the heat equation is:

$$u(x,t) = \frac{12}{\pi} \sum_{n=odd} \frac{e^{-n^2 t}}{n} \sin nx.$$

f) What happens to the solution u as $t \to \infty$? As $t \to \infty$, the terms $e^{-n^2 t}$ go to 0 (in fact, they go to zero faster and faster as n gets bigger). This implies that the solution itself, u, will go to zero as time goes to infinity. This makes sense because, as time passes, the heat will escape from the rod at the two ends, and the temperature of the rod will gradually go down to 0, which is the temperature the two ends are kept at.

Question 4. (10 pts.) Discrete Fourier series.

Let $N = 4$ as we did in class, so that the four points at which we sample our function, between 0 and 2π, are: $0, 2\pi/4 = \pi/2, 2 \times 2\pi/4 = \pi, 3 \times 2\pi/4 = 3\pi/2$.

a) The discrete sine is $\vec{y} = (\sin(0), \sin(\pi/2), \sin(\pi), \sin(3\pi/2)) = (0, 1, 0, -1)$.

b) The Fourier transform of the discrete sine is $(1/4)(1-1, -i-(-i)^3, (-i)^2-(-i)^6, (-i)^3-(-i)^9) = (1/4)(0, -i-i, -1--1, i+i) = (0, -i/2, 0, i/2)$. We can also write this as a sum of complex exponentials:

$$\sum_{k=0}^{3} c_k e^{ikx} = 0e^{i0x} - i/2e^{i1x} + 0e^{i2x} + i/2e^{i3x} = (i/2)(-e^{ix} + e^{i3x}).$$

c) By analogy with the discrete sine, the discrete cosine is $(\cos(0), \cos(\pi/2), \cos(\pi), \cos(3\pi/2)) = (1, 0, -1, 0)$.

d) The Fourier transform of the discrete cosine is $(1/4)(1-1, 1--1, 1-1, 1--1) = (0, 1/2, 0, 1/2)$, which we can rewrite as a sum of complex exponentials:

$$\sum_{k=0}^{3} d_k e^{ikx} = 0e^{i0x} + 1/2e^{i1x} + 0e^{i2x} + 1/2e^{i3x} = 1/2(e^{ix} + e^{i3x}).$$

Question 5. (10 pts.) Odds and ends.

a) Row j of matrix $\bar{F}$ the same as row $N-j$ of matrix F. Entries of row j of matrix $\bar{F}$ are $\bar{w}^{jk} = w^{-jk}$, while entries of row $N-j$ of matrix F are $w^{(N-j)k} = w^{Nk}w^{-jk}$. But $w = e^{i2\pi/N}$ so that $w^N = e^{i2\pi/N*N} = e^{i2\pi} = 1$ so that $w^{Nk} = 1$ as well, and finally $w^{(N-j)k} = w^{-jk}$, as required.

b) If the vector of function values $\vec{y}$ is real, that is, the function values are all real, then its transform $\vec{c}$ has the property that $\bar{c}_{N-k} = c_k$. Indeed, we have that

$$c_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j \bar{w}^{jk} = \frac{1}{N}\sum_{j=0}^{N-1} f_j w^{-jk}$$

and so

$$c_{N-k} = \frac{1}{N}\sum_{j=0}^{N-1} f_j \bar{w}^{j(N-k)}.$$

Taking the complex conjugate, noting that the complex conjugate of N is still N, and same for f_j since f is real, we get

$$\bar{c}_{N-k} = \frac{1}{N}\sum_{j=0}^{N-1} f_j w^{j(N-k)}.$$

By the same argument as in a), $w^{j(N-k)} = w^{-jk}$ so that indeed

$$\bar{c}_{N-k} = \frac{1}{N}\sum_{j=0}^{N-1} f_j w^{-jk} = c_k.$$

Question 6. (10 pts.) Discrete convolution.

Check the cyclic convolution rule directly for $N = 2$: $F(\vec{c} \circledast \vec{d}) = (F\vec{c}). * (F\vec{d})$. Let $\vec{c} = (c_0, c_1)$ and $\vec{d} = (d_0, d_1)$.

To do this, first write F, where $w = e^{i2\pi/N}$ and $N = 2$ so $w = e^{i\pi} = -1$.

$$\begin{pmatrix} 1 & 1 \\ 1 & w \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

so that $F\vec{c} = (c_0 + c_1, c_0 - c_1)^T$ and $F\vec{d} = (d_0 + d_1, d_0 - d_1)^T$, and the product

$$F\vec{c}. * F\vec{d} = \begin{pmatrix} c_0d_0 + c_0d_1 + c_1d_0 + c_1d_1 \\ c_0d_0 - c_0d_1 - c_1d_0 + c_1d_1 \end{pmatrix}.$$

Finally, we obtain

$$\vec{c} \circledast \vec{d} = \begin{pmatrix} c_0d_0 + c_1d_1 \\ c_0d_1 + c_1d_0 \end{pmatrix}$$

and

$$F(\vec{c} \circledast \vec{d}) = \begin{pmatrix} c_0d_0 + c_1d_1 + c_0d_1 + c_1d_0 \\ c_0d_0 + c_1d_1 - c_0d_1 - c_1d_0 \end{pmatrix}.$$

Thus we see that, indeed, the cyclic convolution rule holds and $F\vec{c}. * F\vec{d} = F(\vec{c} \circledast \vec{d})$.