18.085, PROBLEM SET 8

DUE DURING THE "MONDAY 8/11" CLASS, STILL TO BE SCHEDULED

Question 1. (15 pts.) Question 4 of pset7, continued.

Let $f(x)$ be the 2π-periodic extension of the function $g(x) = e^{-x}$ defined on the interval $-\pi \le x \le \pi$. See the solutions to pset7 online for the answers to a) (sketch of f) and b) the complex Fourier series for f.

c) Find the Fourier series of df/dx from the Fourier series of f.

d) Now find the Fourier series of the sum $f' + f$. Note that $g' + g = 0$, so explain why your series for $f' + f$ isn't 0?

Hint: Recall the Fourier series for the delta function $\delta(x)$ derived in class:

$$\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\left(\cos x + \cos 2x + \cos 3x + \dots\right).$$

Question 2. (25 pts.) Decay and derivatives.

Let $f(x) = x(\pi - x)$ over $0 \le x \le \pi$, and let g be its odd extension, that is, $g(x) = f(x)$ for $x \in [0, \pi]$ and $g(x) = -f(-x)$ for $x \in [-\pi, 0]$. We shall investigate g, and its first two derivatives, knowing that the Fourier series of g, an odd function, is:

$$g(x) = \frac{8}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots\right).$$

a) (Use Matlab for the plot.) In one plot, over the interval $x \in [\pi/3, 2\pi/3]$ only, show the function g and its first three partial sums. As you take more and more terms in the partial sums, do you observe the behavior you would expect?

b) (By hand.) What is the first derivative in x of g? What is the first derivative in x of its Fourier series?

c) Same as in (a), but with the first derivative of g as found in (b). Don't forget to comment on the behavior of the partial sums.

d) What is the second derivative in x of g? What is the second derivative in x of its Fourier series?

e) Same as in (a), but with the second derivative of g as found in (d). Don't forget to comment on the behavior of the partial sums.

f) What is the decay rate of the Fourier coefficients of g? dg/dx? d^2g/dx^2? Explain your answer using the continuity of those functions and their derivatives.

g) Rewrite the Fourier series you found in (d) in terms of the Fourier series of a function we've seen in class. Hint: look at the plot of d^2g/dx^2 over $[-\pi, \pi]$.

Question 3. (30 pts.) The Heat Equation.

If you find this question confusing, you can follow along with the worked example in the book, at the end of section 4.1 (pages 329-331).

We consider the temperature $u = u(x,t)$ in a rod, over the interval $x \in [0,\pi]$ and for time $t > 0$. This temperature must satisfy the heat equation:

$$u_t = u_{xx}, \ x \in [0,\pi], \ t > 0. \tag{0.1}$$

At time $t = 0$, we are given that the temperature is uniform and of value 3 (initial condition):

$$u(x,0) = 3, \ x \in [0,\pi]. \tag{0.2}$$

The rod is coated so that heat can only escape at the ends. At time $t = 0$, the rod is moved into a freezer (zero temperature). This will cause the ends of the rod to be at temperature 0 for all positive times (boundary conditions):

$$u(0,t) = u(\pi,t) = 0, \ t > 0. \tag{0.3}$$

In this question, we solve the heat equation using Fourier series. The boundary conditions (0.3) of 0 imply we should use a sine series (the sines will be automatically 0 where they should). This means we need the initial condition to be odd (so it has a sines series).

a) Sketch the odd extension of the function $f(x) = 3$ from the interval $x \in [0,\pi]$ to the interval to $x \in [-\pi,\pi]$.
b) Find the sine Fourier series of the odd extension in a) (you should be able to write this in terms of a known Fourier series, instead of having to calculate each coefficient).

Now we write the solution u (unknown at this point) in a sine series as well, but with coefficients (to be found) that vary in time:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx. \tag{0.4}$$

Notice how now the solution u in the interval $x \in [-\pi,0]$ is the odd extension of u in the interval $x \in [0,\pi]$. Just like we did with the initial condition.

c) Plug in the solution (0.4) into the heat equation (0.1), to obtain a differential equation that the b_n's must satisfy.
d) Solve the differential equation you obtained in (c), assuming initial conditions $b_n(0)$ for each n.

All that remains is to find the actual initial conditions $b_n(0)$! We know u must satisfy the initial condition (0.2), and we have sine Fourier series for each.

e) Match the terms in the Fourier series of u and the initial condition to obtain the value of the $b_n(0)$, for each n.
f) What happens to the solution u as $t \to \infty$? (You don't need e) to answer this.)

To verify your answer, you can think about what you expect the solution to do physically in this setting.

Question 4. (10 pts.) Discrete Fourier series.

Let $N = 4$ as we did in class, so that the four points at which we sample our function, between 0 and 2π, are: $0, 2\pi/4 = \pi/2, 2 \times 2\pi/4 = \pi, 3 \times 2\pi/4 = 3\pi/2$.

a) Let $f(x) = \sin x$ over $0 \leq x \leq 2\pi$. Sample this function at our four points in x to get its vector in x-space $\vec{y}$: $\vec{y} = (f(0), f(\pi/2), \dots)$. This is called the discrete sine.
b) What is the Fourier transform of the discrete sine? Write your answer as a vector with 4 entries, and also as a sum of complex exponentials.
c) By analogy with the discrete sine, what is the discrete cosine?
d) What is the Fourier transform of the discrete cosine? Write your answer as a vector with 4 entries, and also as a sum of complex exponentials.

Question 5. (10 pts.) Odds and ends.

a) Why is row j of matrix $\bar{F}$ the same as row $N - j$ of matrix F? (You should check this for any N, and any $0 \leq j \leq N - 1$.)
b) If the vector of function values $\vec{y}$ is real, that is, the function values are all real, then show that its transform $\vec{c}$ has the property that $\bar{c}_{N-k} = c_k$. (You should check this for any N, and any $0 \leq k \leq N - 1$.)

Question 6. (10 pts.) Discrete convolution.

Check the cyclic convolution rule directly for $N = 2$: $F(c \circledast d) = (Fc) .* (Fd)$. Let $\vec{c} = (c_0, c_1)$ and $\vec{d} = (d_0, d_1)$.

To do this, first write F. Then calculate $F\vec{c}$ and $F\vec{d}$, and the product $F\vec{c} .* F\vec{d}$. Finally, obtain $\vec{c} \circledast \vec{d}$ and $F(\vec{c} \circledast \vec{d})$.