18.085: PROBLEM SET 7

Due August 4, 2014 9:30AM IN CLASS

Question 1. (35 pts.) Solving Poisson's equation numerically.

We will solve Poisson's equation

$$-\Delta u = 6(x-y)+2, \quad 0 \le x \le 1, 0 \le y \le 1$$

on the unit square, subject to the boundary conditions

- $u = x^2(1-x)$ on the horizontal edges of the square, and
- $u = y(1-y)^2$ on the vertical edges.

(a) Verify that $u(x,y) = x^2(1-x) + y(1-y)^2$ is the exact solution to this problem. To visualize this solution in Matlab, use the following few lines of code and print the plot:

```
[X,Y] = meshgrid(0:.001:1);
U_exact = (X.^2).*(1-X) + Y.*((1-Y).^2);
surf(X,Y,U_exact,'edgecolor','none');
view([0,90]); colorbar; axis square;
```

(b) We will now solve the equation numerically in Matlab. Let N be the number of unknowns in each direction (x and y). Number the nodes by row from bottom to top, as we did in class. Write a code that follows these steps:

i. Construct the matrix K, the 1D second-derivative matrix, using the following few lines of code:

```
v = [2 -1 zeros(1,N-2)];
K = toeplitz(v);
```

ii. Construct the matrix $K2D$, the 5-point Laplacian matrix in 2D, using K. *Hint:* Equation (3) on page 284 of the textbook is useful.

iii. Now construct the vector $\vec{F}$ using the right-hand side of Poisson's equation, ignoring the boundary conditions for now. (Careful with the indexing!)

iv. As we saw in class, the boundary conditions modify some of the entries of $\vec{F}$. For now, ignore the four corners; we will deal with them in the next step. For example, we can modify $\vec{F}$ using the boundary conditions on the bottom edge as follows:

```
for k = 2:N-1
    F(k) = F(k) + g(k*h,0);
end;
```

In this code, $g(x, y)$ is the boundary value of $u(x, y)$, and h is the grid spacing. Do the same for the other edges. (Careful on how h relates to N.)

v. Now deal with the four corners. For example, the value of $\vec{F}$ corresponding to the bottom left corner is modified as follows:

```
F(1) = F(1) + g(h,0) + g(0,h);
```

Do the same for the other three corners.

vi. Solve for the unknowns $\vec{U}$ using backslash.

vii. To visualize your solution, you will need to rearrange the components of your vector $\vec{U}$ in matrix form. To do so, use the following few lines of code:

```
UMat = reshape(U,N,N)';
[X2,Y2] = meshgrid(h:h:1-h);
surf(X2,Y2,UMat,'edgecolor','none');
view([0,90]); axis square; colorbar;
```

(c) Show plots of your approximate solutions for $N = 10$, 30 and 50. How do they compare (visually) to the exact solution you found in part (a)?

Question 2. (15 pts.) 9-point stencil for the Laplacian.

In class, we showed that the 5-point approximation for the negative Laplacian $-\Delta$ is given by

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \approx \frac{1}{h^2}\left(4u(x,y) - u(x+h,y) - u(x-h,y) - u(x,y+h) - u(x,y-h)\right)$$

where h is the grid spacing. This approximation is accurate to order $O(h^2)$. It turns out that the following 9-point approximation to the Laplacian may be more accurate:

$$\begin{aligned}-\Delta u \quad \approx \quad & \frac{1}{6h^2}\left[20u(x,y) - 4\left(u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)\right)\right. \\ & \left. - \left(u(x+h,y+h) + u(x+h,y-h) + u(x-h,y+h) + u(x-h,y-h)\right)\right]\end{aligned}$$

Suppose you want to solve Poisson's equation $-\Delta u = f$ on the unit square ($0 \le x \le 1$, $0 \le y \le 1$) with some boundary conditions. Write down the matrix K corresponding to the 9-point Laplacian for $h = 1/4$.

Question 3. (30 pts.) Fourier series of $f(x) = x$.

Let $f(x)$ be the 2π-periodic extension of the function $g(x) = x$ defined on the interval $-\pi \le x < \pi$. The function $f(x)$ is called the *sawtooth wave.*

(a) Sketch $f(x)$.

(b) Find the Fourier series for $f(x)$. Write the series in terms of sines and cosines. You can either find the sine/cosine series directly or deduce it from the complex series.

Hint: Use integration by parts.

(c) Use Parseval's theorem to deduce the following identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Question 4. (20 pts.) Fourier series of $f(x) = \mathrm{e}^{-x}$.

Let $f(x)$ be the 2π-periodic extension of the function $g(x) = \mathrm{e}^{-x}$ defined on the interval $-\pi \le x < \pi$.

(a) Sketch $f(x)$.

(b) Find the complex Fourier series for $f(x)$.

Bonus (5 pts.). Fourier coefficients and differentiability

(a) Let $f_2(x)$ be the periodic extension of the function $g_2(x) = x^2$ defined on the interval $-1 \le x < 1$. Show that $f_2(x)$ is continuous, but that f_2' is not continuous.

(b) Let $f_3(x)$ be the periodic extension of the function $g_3(x) = x^3 + ax^2 + bx$ defined on the same interval. Find the constants a and b such that f_3 and f_3' are both continuous. Show that f_3'' is not continuous.

(c) Let $f_4(x)$ be the periodic extension of the function $g_4(x) = x^4 + ax^3 + bx^2 + cx$ defined on the same interval. Find the constants a, b and c such that f_4, f_4' and f_4'' are continuous. Show that f_4''' is not continuous.

(d) Find the Fourier coefficients of f_2, f_3 and f_4 in Matlab. Show plots demonstrating that they decay (in absolute value) like $1/n^2$, $1/n^3$ and $1/n^4$, respectively. Explain why this is the case.

(e) Let $f(x) = \mathrm{e}^{\cos(\pi x)}$. Find the Fourier coefficients of f in Matlab, and show a plot demonstrating that they decay exponentially in n. Explain why this is the case.

Hints for parts (d) and (e):

- You can find the integral of a function (for example, $f(x) = x^2$) on the interval $[a, b]$ in Matlab using the following command:

  `quadgk(@(x) x.^2,a,b)`

  Note the dot after x.
- To demonstrate the rate of decay of the Fourier coefficients, you may want to think about plotting their log ...