18.085: PROBLEM SET 6 SOLUTIONS

Question 1. (10 pts.) Cubic finite elements.

Recall that essential boundary conditions are imposed on the finite elements, but natural boundary conditions are not. For the equation $u'' = f(x)$ (elastic bar), boundary conditions on u are essential, while those on u' are natural. For the equation $u'''' = f(x)$ (bending beam), boundary conditions on u and u' are essential, whereas those on u'' and u''' are natural.

(a) The boundary conditions on u and u' are essential, so they must be incorporated. We drop ϕ_0^d and ϕ_3^d because they do not satisfy $\phi(0) = 0$ and $\phi(1) = 0$, respectively. We also drop ϕ_0^s and ϕ_3^s because they do not satisfy $\phi'(0) = 0$ and $\phi'(1) = 0$, respectively. Indeed, it is clear from the definitions of the cubic finite elements (and their graphs on page 245 of the textbook) that $\phi_n^d = 1$ and $(\phi_n^s)' = 1/h$ at the meshpoint $x = nh$.

(b) The boundary conditions on u are both essential, so they must be incorporated. We drop ϕ_0^d and ϕ_3^d because they do not satisfy $\phi(0) = 0$ and $\phi(1) = 0$, respectively.

(c) The boundary condition on u is essential, but the one on u' is not. We thus only drop ϕ_0^d, because it does not satisfy $\phi(0) = 0$.

(d) The boundary conditions on u are essential, but those on u'' is not. We thus only drop ϕ_0^d and ϕ_3^d because they do not satisfy $\phi(0) = 0$ and $\phi(1) = 0$, respectively.

Question 2. (20 pts.) Laplace's equation and level curves.

(a) The equation is

$$u_{xx} + u_{yy} = y^2 - 3\lambda x - 1$$

To find a solution, you can integrate the x-terms twice with respect to x, and the y-terms twice with respect to y. (You can do either for the constant term). This works because each term is a function either x or y but not both. This yields the solution

$$u(x, y) = \frac{y^4}{12} - \frac{\lambda x^3}{2} - \frac{x^2}{2}$$

(b) Recall that the solutions to Laplace's equation in polar coordinates have the form $u(r,\theta) = r^n \cos n\theta$ and $u(r,\theta) = r^n \sin n\theta$. We simply need to pick out the values of n that satisfy the boundary condition $u(1,\theta) = \cos 2\theta + \sin 6\theta + 1$. The solution thus has the form

$$u(r,\theta) = r^2 \cos 2\theta + r^6 \sin 6\theta + 1$$

(c) This equation does not have a solution. Note that the left side of the equation is

$$\frac{\partial}{\partial x}\left(-\frac{\partial s}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial s}{\partial x}\right) = -\frac{\partial^2 s}{\partial x \partial y} + \frac{\partial^2 s}{\partial y \partial x} = 0$$

which cannot equal the right side e^{x^2-3y}.

(d) The gradient of $u(x,y)$ is

$$\nabla u = (\mathrm{e}^{-y}\cos x, -\mathrm{e}^{-y}\sin x)$$

Note that the vector field ∇u is divergence-free:

$$\mathrm{div}(\nabla u) = -\mathrm{e}^{-y}\sin x + \mathrm{e}^{-y}\sin x = 0$$

The vector field ∇u thus admits a stream function $S(x,y)$ that satisfies

$$\mathrm{e}^{-y}\cos x = \frac{\partial S}{\partial y}, \quad -\mathrm{e}^{-y}\sin x = -\frac{\partial S}{\partial x}$$

Recall from class that the streamlines (the level curves of the stream function $S(x,y)$) are perpendicular to the level curves of $u(x,y)$. We thus need to find the function S. Integrating both sides of the equations above, we have

$$S(x,y) = -\mathrm{e}^{-y}\cos x + c(x), \quad S(x,y) = -\mathrm{e}^{-y}\cos x + d(y)$$

We thus conclude that the level curves of $S(x,y) = -\mathrm{e}^{-y}\cos x$ are perpendicular to those of $u(x,y) = \mathrm{e}^{-y}\sin x$.

(e) **Bonus:** Recall that $\mathrm{e}^{iz} = \mathrm{e}^{ix}\mathrm{e}^{-y} = \mathrm{e}^{-y}(\cos x + i\sin x)$, where $z = x + iy$. Or, $-i i\mathrm{e}^{iz} = \mathrm{e}^{-y}(\sin x - i\cos x)$. We thus conclude that

$$u(x,y) = \mathrm{e}^{-y}\sin x = \Re(-i\mathrm{e}^{iz}), \quad S(x,y) = -\mathrm{e}^{-y}\cos x = \Im(-i\mathrm{e}^{iz})$$

so the relevant function is $f(z) = -i\mathrm{e}^{iz}$.

Question 3. (20 pts) Laplace's equation on a square.

(a) Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \frac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) = \cosh x, \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sinh x = \frac{1}{2}\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right) = \sinh x$$

We also know that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin x = -\sin x$$

Let $u_n(x,y) = \sin(\pi n x)\sinh(\pi n y)$. Using the chain rule, we find that

$$\frac{\partial^2 u_n}{\partial x^2} = -(\pi n)^2 \sin(n\pi x)\sinh(n\pi y), \quad \frac{\partial^2 u_n}{\partial y^2} = (\pi n)^2 \sin(n\pi x)\sinh(n\pi y)$$

Combining these expressions, we conclude that

$$\Delta u_n = \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} = -(\pi n)^2 \sin(\pi n x)\sinh(\pi n y) + (\pi n)^2 \sin(\pi n x)\sinh(\pi n y) = 0$$

as desired.

(b) We impose the boundary conditions $u = 0$ at $x = 0$ and $x = 1$. The condition at $x = 0$ is automatically satisfied (since $\sin(0) = 0$). For the condition at $x = 1$ to be satisfied, we need $\sin(n\pi) = 0$, which implies that n must be an integer.

(c) Note that the function $u_3(x,y) = \sin(3\pi x)\sinh(3\pi y)$ satisfies the boundary condition $u = 0$ on every edge of the square except the top one ($x = 0$, $x = 1$, $y = 0$). We see that the value on the top edge $y = 1$ is $u_3(x,1) = \sin(3\pi x)(\sinh 3\pi)$, which is correct apart from the constant factor $\sinh(3\pi)$. The solution is thus

$$u(x,y) = \frac{\sin(3\pi x)\sinh(3\pi y)}{\sinh(3\pi)}$$

(d) For $u_n(x,y) = \sin(n\pi x)\sinh(\pi n(1-y))$, the derivatives with respect to x are the same as those in part (a). The derivatives with respect to y are

$$\frac{\partial u_n}{\partial y} = -\pi n \sin(n\pi x)\cosh(\pi n(1-y)), \quad \frac{\partial^2 u_n}{\partial y^2} = (\pi n)^2 \sin(n\pi x)\sinh(\pi n(1-y))$$

We thus conclude that

$$\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} = -(\pi n)^2 \sin(n\pi x)\sinh(\pi n(1-y)) + (\pi n)^2 \sin(n\pi x)\sinh(\pi n(1-y)) = 0$$

as desired. Note that the solution $u_5(x,y) = \sin(5\pi x)\sinh(5\pi(1-y))$ satisfies the boundary conditions $u = 0$ on each edge of the square except the bottom one ($y = 0$). Since $u_5(x,0) = \sin(5\pi x)\sinh(5\pi)$, we find that the solution is

$$u(x,y) = \frac{\sin(5\pi x)\sinh(5\pi(1-y))}{\sinh(5\pi)}$$

(e) **Bonus:** We simply need to piece together the solutions defined in parts (a) and (d). Note that the functions

$$\begin{aligned} U_1(x,y) &= \frac{\sin(n_1\pi x)\sinh(n_1\pi(1-y))}{\sinh(n_1\pi)} \\ U_2(x,y) &= \frac{\sinh(n_2\pi x)\sin(n_2\pi y)}{\sinh(n_2\pi)} \\ U_3(x,y) &= \frac{\sin(n_3\pi x)\sinh(n_3\pi y)}{\sinh(n_3\pi)} \\ U_4(x,y) &= \frac{\sinh(n_4\pi(1-x))\sin(n_4\pi y)}{\sinh(n_4\pi)} \end{aligned}$$

all satisfy Laplace's equation $\Delta u = 0$, and are identically zero on three edges of the unit square. For example, $U_1 = 0$ on $x = 0$, $x = 1$ and $y = 1$, and $U_2 = 0$ on $x = 0$, $y = 0$ and $y = 1$. In addition, it is clear that

$$U_1(x,0) = \sin(n_1\pi x), \quad U_2(1,y) = \sin(n_2\pi y), \quad U_3(x,1) = \sin(n_3\pi x), \quad U_4(0,y) = \sin(n_4\pi y)$$

so the functions individually satisfy the appropriate boundary conditions on each of the four edges of the square. Since Laplace's equation is linear, the complete solution is

$$u(x,y) = U_1(x,y) + U_2(x,y) + U_3(x,y) + U_4(x,y).$$