18.085, PROBLEM SET 6, DUE 7/25 (IN CLASS)

Question 1. (10 pts.) Cubic finite elements.

Consider the cubic finite elements $\phi_i^d(x)$ and $\phi_i^s(x)$ (where $i = 0, 1, 2, 3$) based at the meshpoints $x = 0, 1/3, 2/3, 1$. Which functions ϕ are dropped because of the given differential equations and associated boundary conditions? Explain your answers.

(a) $u'''' = f(x)$ with boundary conditions $u = u' = 0$ at the ends $x = 0$ and $x = 1$.

(b) $-u'' = f(x)$ with boundary conditions $u(0) = u(1) = 0$.

(c) $-u'' = f(x)$ with boundary conditions $u(0) = u'(1) = 0$.

(d) $u'''' = f(x)$ with boundary conditions $u = u'' = 0$ at the ends $x = 0$ and $x = 1$.

Hint: Recall that boundary conditions on u are *essential (Dirichlet)* and those on u' are *natural (Neumann)* for the elastic bar. For the bending beam, boundary conditions on u and u' are *essential* and those on higher derivatives are *natural*.

Question 2. (20 pts.) Laplace's equation and level curves.

Determine if the equations in parts (a)–(c) can be solved. Write a solution if one exists, and show your work. Otherwise, explain why it is not possible to find a solution.

(a)

$$\operatorname{div}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = y^2 - 3\lambda x - 1$$

(b) $\Delta u = 0$ on the unit disc with boundary conditions $u(1, \theta) = \cos 2\theta + \sin 6\theta + 1$ on the boundary of the disc.

(c)

$$\operatorname{div}\begin{pmatrix} -\frac{\partial s}{\partial y} \\ \frac{\partial s}{\partial x} \end{pmatrix} = \mathrm{e}^{x^2 - 3y}$$

(d) Find a family of curves $S(x, y) = C$ that is everywhere orthogonal (perpendicular) to the level curves of $u(x, y) = \mathrm{e}^{-y} \sin x$.

(e) **Bonus (5 pts.):** The function $u(x, y) = \mathrm{e}^{-y} \sin x$ and $S(x, y)$ that you found in part (d) are the real and imaginary parts of a complex function. What is it?

Question 3. (20 pts) Laplace's equation on a square.

In this question, we solve Laplace's equation $\Delta u = 0$ on the unit square $R = [0, 1] \times [0, 1] = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.

(a) Verify that $u_n(x, y) = \sin(\pi n x) \sinh(\pi n y)$ solves Laplace's equation $\Delta u = 0$ for any real number n. (Hint: Recall that $\sinh x = \left(\mathrm{e}^x - \mathrm{e}^{-x}\right)/2$.)

(b) We now impose the boundary condition $u = 0$ on the left and right edges of the square. For what values of n is $u_n(x, y)$ a solution?

(c) Solve Laplace's equation $\Delta u = 0$ with the boundary condition $u = \sin 3\pi x$ on the top edge of the square, and $u = 0$ on the three remaining edges.

(d) Verify that $u_n(x, y) = \sin(\pi n x) \sinh(\pi n(1-y))$ also solves Laplace's equation $\Delta u = 0$. Then solve Laplace's equation with the boundary condition $u = \sin 5\pi x$ on the bottom edge of the square, and $u = 0$ on the three remaining edges.

(e) **Bonus (5 pts.):** Write the solution to Laplace's equation $\Delta u = 0$ on the unit square with boundary conditions $u = \sin(n_1 \pi x)$ on the lower edge, $u = \sin(n_2 \pi y)$ on the right edge, $u = \sin(n_3 \pi x)$ on the upper edge, and $u = \sin(n_4 \pi y)$ on the left edge, where n_i are integers.