18.085: PROBLEM SET 5 SOLUTIONS

Question 1. (15 pts.) A hanging bar.

(a) w solves the equation

$$-\frac{\mathrm{d}w}{\mathrm{d}x} = \delta(x - 1/2), \quad w(1) = 0 \tag{0.1}$$

As shown in class, the solution is

$$w(x) = \int_x^1 \delta(s - 1/2)\,\mathrm{d}s = \begin{cases} 1 & \text{if } x < 1/2 \\ 0 & \text{if } x > 1/2 \end{cases} \tag{0.2}$$

So $w(x)$ is a step function, as shown in Figure 1.

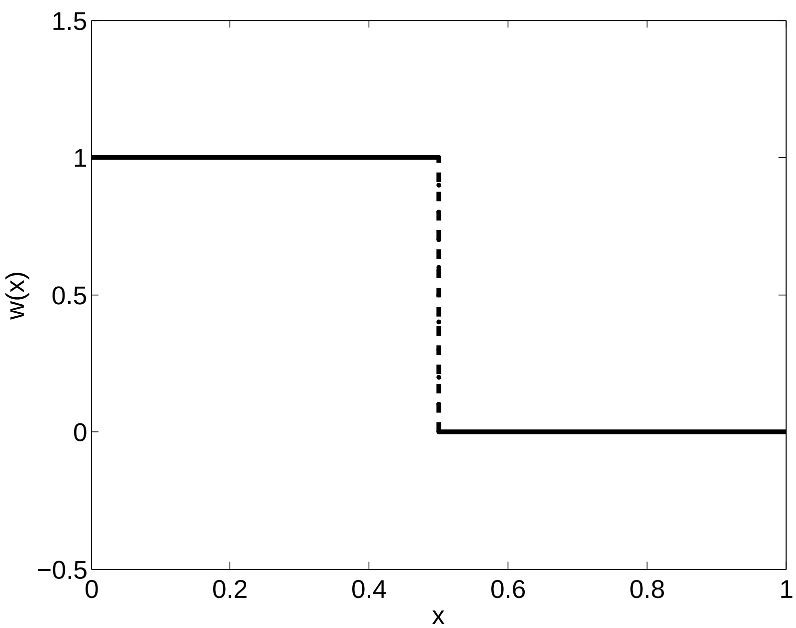

FIGURE 1. Solution to Question 1(a).

If you don't want to use the solution provided in class, you can directly integrate both sides of equation (0.1):

$$w(x) = -S(x - 1/2) + C$$

where S is a step function, and C is a constant to be determined. We now use the boundary condition $w(1) = 0$ to get $C = 1$, so the solution is

$$w(x) = 1 - S(x - 1/2)$$

This is equivalent to the solution (0.2).

(b) Since $w(x) = c(x)u'(x)$, we use part (a) to find that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{w(x)}{c(x)} = \begin{cases} 2 & \text{if } x < 1/2 \\ 0 & \text{if } x > 1/2 \end{cases}$$

This is shown in Figure 2.

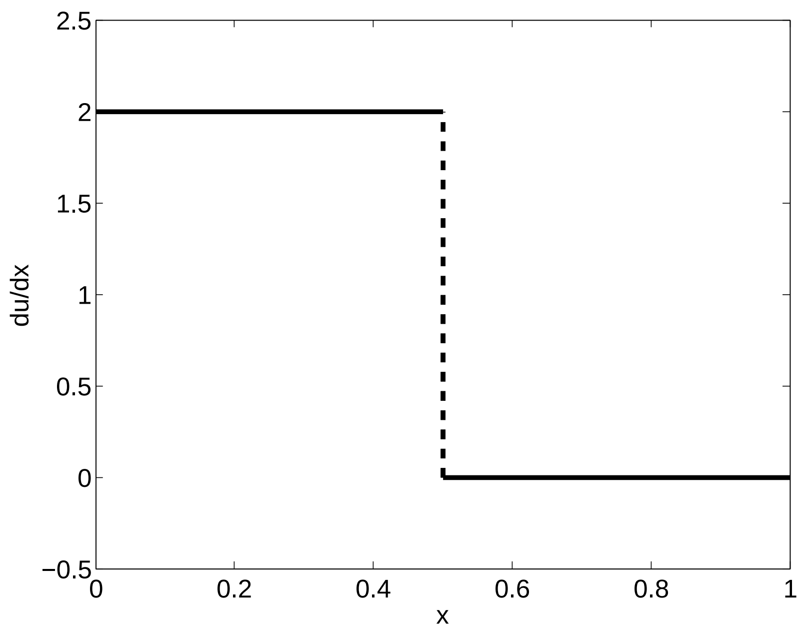

FIGURE 2. Solution to Question 1(b).

(c) We can use the formula for $u(x)$ provided in class:

$$u(x) = \int_0^x \frac{w(s)}{c(s)}\,\mathrm{d}s$$

We found the integrand $w(s)/c(s)$ in part (b). If $x < 1/2$, we have

$$u(x) = \int_0^x \frac{w(s)}{c(s)}\, \mathrm{d}s = \int_0^x 2\, \mathrm{d}s = 2x$$

If $x > 1/2$,we have

$$u(x) = \int_0^x \frac{w(s)}{c(s)}\, \mathrm{d}s = \int_0^{1/2} 2\, \mathrm{d}s + \int_{1/2}^x 0\, \mathrm{d}s = 1$$

Combining these answers, we obtain the solution

$$u(x) = \begin{cases} 2x & \text{if } x < 1/2 \\ 1 & \text{if } x > 1/2 \end{cases} \tag{0.3}$$

This is shown in Figure 3.

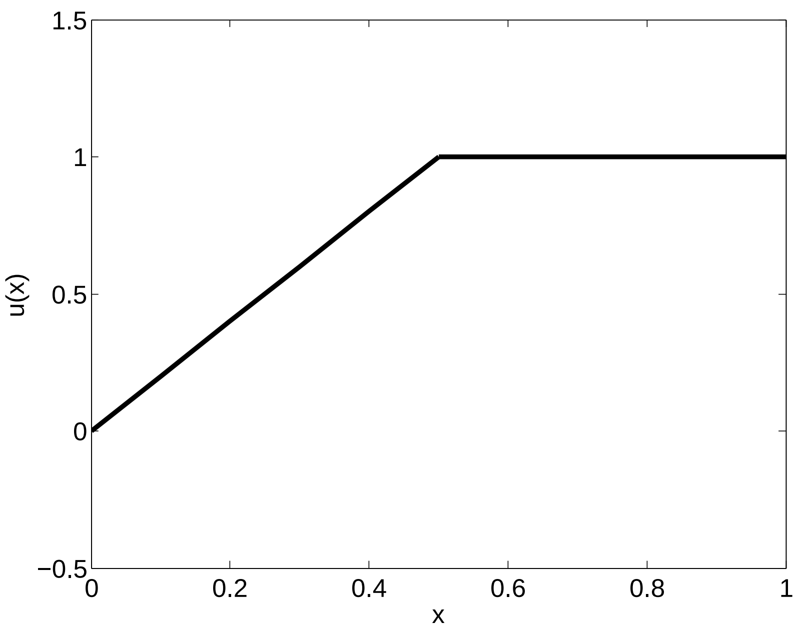

FIGURE 3. Solution to Question 1(c).

Question 2. (35 pts.) Finite element method with hat functions.

(a) We need two hat functions $\phi_1(x)$ and $\phi_2(x)$. We do not need the half-hats, since we have zero boundary conditions $u(0) = u(1) = 0$ at each end. The hat functions are

$$\phi_1(x) = \begin{cases} 3x & \text{if } 0 \le x \le 1/3, \\ 2 - 3x & \text{if } 1/3 \le x \le 2/3, \\ 0 & \text{otherwise} \end{cases}, \quad \phi_2(x) = \begin{cases} 3x - 1 & \text{if } 1/3 \le x \le 2/3, \\ 3 - 3x & \text{if } 2/3 \le x \le 1, \\ 0 & \text{otherwise} \end{cases}$$

The derivatives are

$$\phi_1'(x) = \begin{cases} 3 & \text{if } 0 < x < 1/3, \\ -3 & \text{if } 1/3 < x < 2/3, \\ 0 & \text{otherwise} \end{cases}, \quad \phi_2'(x) = \begin{cases} 3 & \text{if } 1/3 < x < 2/3, \\ -3 & \text{if } 2/3 < x < 1, \\ 0 & \text{otherwise} \end{cases}$$

Plots of the functions are shown in Figure 4, and their derivatives in Figure 5.

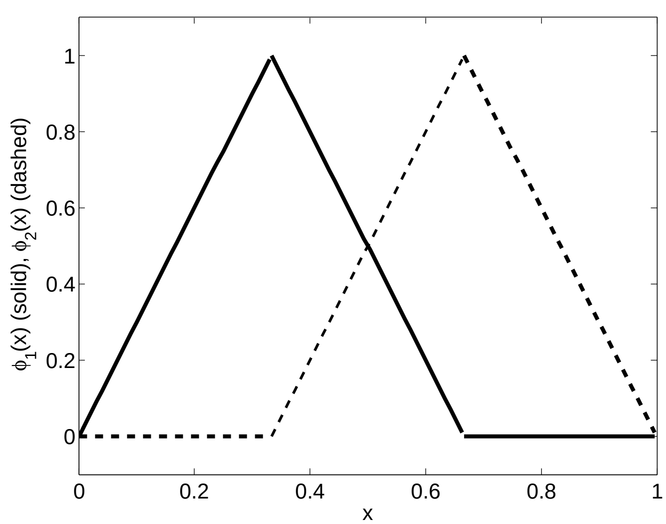

FIGURE 4. Plots of the hat functions $\phi_1(x)$ (solid) and $\phi_2(x)$ (dashed) for Question 2(a).

(b) We now find the matrix K. Recall that

$$K_{ij} = \int_0^1 \phi_i'(x)\phi_j'(x)\,\mathrm{d}x$$

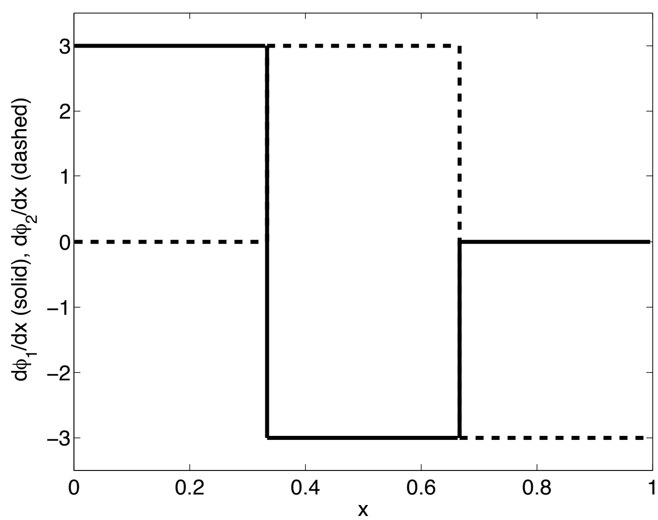

FIGURE 5. Plots of the derivatives of the hat functions $\phi_1(x)$ (solid) and $\phi_2(x)$ (dashed) for Question 2(a).

Since the derivatives of $\phi_i(x)$ are constants on each interval, the integrals aren't too difficult:

$$K_{11} = \int_0^1 \phi_1'(x)^2 \, \mathrm{d}x = \int_0^{1/3} 3^2 \, \mathrm{d}x + \int_{1/3}^{2/3} (-3)^2 \, \mathrm{d}x = 6$$

It is clear that $K_{22} = 6$ as well. Moving on:

$$K_{12} = \int_0^1 \phi_1'(x)\phi_2'(x) \, \mathrm{d}x = \int_0^{1/3} 3 \cdot 0 \, \mathrm{d}x + \int_{1/3}^{2/3} 3 \cdot (-3) \, \mathrm{d}x + \int_{2/3}^{1} (-3) \cdot 0 \, \mathrm{d}x = -3$$

Since K is symmetric, $K_{21} = -3$ as well. The matrix K thus looks like

$$K = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$$

(c) Recall that the elements of the vector $\vec{F}$ are given by the formula

$$F_i = \int_0^1 f(x)\phi_i(x) \, \mathrm{d}x$$

Beginning with $\phi_1(x)$, we have

$$F_1 = \int_0^1 x\phi_1(x)\,\mathrm{d}x = \int_0^{1/3} x \cdot 3x\,\mathrm{d}x + \int_{1/3}^{2/3} x\cdot(2-3x)\,\mathrm{d}x = \frac{1}{3}\cdot\frac{1}{6}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{9}$$

$$F_2 = \int_0^1 x\phi_2(x)\,\mathrm{d}x = \int_{1/3}^{2/3} x\cdot(3x-1)\,\mathrm{d}x + \int_{2/3}^{1} x\cdot(3-3x)\,\mathrm{d}x = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{5}{6}\cdot\frac{1}{2} = \frac{2}{9}$$

where we use the midpoint rule to do each integral. The vector $\vec{F}$ is thus

$$\vec{F} = \begin{pmatrix} 1/9 \\ 2/9 \end{pmatrix}$$

(d) We now solve the system $K\vec{U} = \vec{F}$:

$$3\cdot\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 1/9 \\ 2/9 \end{pmatrix} \Rightarrow \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 1/27 \\ 2/27 \end{pmatrix}$$

We now use the formula for the inverse of a 2×2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

to obtain

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1/27 \\ 2/27 \end{pmatrix} = \begin{pmatrix} 4/81 \\ 5/81 \end{pmatrix}$$

Our solution is thus

$$U(x) = \frac{4}{81}\phi_1(x) + \frac{5}{81}\phi_2(x) \tag{0.4}$$

A plot of this function is in Figure 6, together with the exact solution (which we derive in part (d)). The code I used to make this plot may be found at the end of this document.

(e) We now find the exact solution to the equation $u'' = -x$ with boundary conditions $u(0) = u(1) = 0$. Integrating both sides, we obtain

$$u' = -\frac{x^2}{2} + C$$

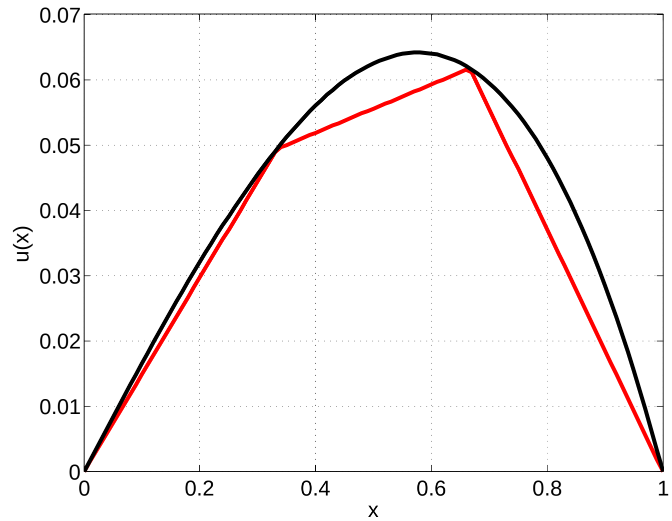

FIGURE 6. Red curve shows the approximate solution $U(x)$ derived in Question 2(d), and the black curve shows the exact solution $u(x)$ derived in Question 2(e).

Integrating again, we obtain

$$u(x) = -\frac{x^3}{6} + Cx + D$$

The condition $u(0) = 0$ implies that $D = 0$, and $u(1) = 0$ implies that $C = 1/6$. We thus obtain the exact solution

$$u(x) = \frac{x - x^3}{6}$$

Note that the approximate solution $U(x)$ defined in (0.4) has the values $U(1/3) = 4/81$ and $U(2/3) = 5/81$ at the node points. It is easy to see that the exact solution assumes the same values: $u(1/3) = 8/(27 \cdot 6) = 4/81$ and $u(2/3) = 10/(27 \cdot 6) = 5/81$.

To approximate the error, we let $\mathtt{u}$ be the vector corresponding to the exact solution and $\mathtt{U}$ the vector corresponding to the approximate solution. As stated in the e-mail hint, we use the command $[\mathtt{a}, \mathtt{b}] = \mathtt{max}(\mathtt{abs}(\mathtt{u} - \mathtt{U}))$ to determine the size of the maximum error ($\mathtt{a}$) and its location ($\mathtt{xv}(\mathtt{b})$). Using the vector of x-values $\mathtt{xv} = \mathtt{0} : \mathtt{.01} : \mathtt{1}$, we find that the maximum error is 0.0016 at $x = 0.84$.

Bonus: To find the precise answer, we define the function $E(x) = U(x) - u(x)$. Note that the approximate solution $U(x)$ found in part (d) has the form

$$U(x) = \begin{cases} \frac{4}{27}x & \text{if } 0 \le x \le 1/3 \\ \frac{x}{27} + \frac{3}{81} & \text{if } 1/3 \le x \le 2/3 \\ -\frac{5}{27}x + \frac{5}{27} & \text{if } 2/3 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We thus find that

$$E(x) = \begin{cases} -x/54 + x^3/6 & \text{if } 0 \le x \le 1/3 \\ -7x/54 + 3/81 + x^3/6 & \text{if } 1/3 \le x \le 2/3 \\ -19x/54 + 5/27 + x^3/6 & \text{if } 2/3 \le x \le 1 \end{cases}$$

To find the maximum, we take the derivative of $E(x)$:

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \begin{cases} -1/54 + x^2/2 & \text{if } 0 \le x \le 1/3 \\ -7/54 + x^2/2 & \text{if } 1/3 \le x \le 2/3 \\ -19/54 + x^2/2 & \text{if } 2/3 \le x \le 1 \end{cases}$$

The extrema of $E(x)$ correspond to the zeros of dE/dx:

$$\begin{aligned} -1/54 + x^2/2 &= 0 \Rightarrow x = 1/\sqrt{27} \\ -7/54 + x^2/2 &= 0 \Rightarrow x = \sqrt{7/27} \\ -19/54 + x^2/2 &= 0 \Rightarrow x = \sqrt{19/27} \end{aligned}$$

Substituting these values back into the function $E(x)$, we find

$$E(1/\sqrt{27}) \approx -0.0024, \quad E(\sqrt{7/27}) \approx -0.0070, \quad E(\sqrt{19/27}) \approx -0.0116$$

We thus find that the maximum error occurs at $x = \sqrt{19/27} \approx 0.8389$, and that it has the value $|E(\sqrt{19/27})| \approx 0.0116$.

Question 3. (35 pts.) Finite element method with bubble functions.

(a) We will use three bubble functions $\phi_3(x)$, $\phi_4(x)$ and $\phi_5(x)$, defined as

$$\phi_3(x) = \begin{cases} -36x(x-1/3) & \text{if } 0 \le x \le 1/3 \\ 0 & \text{otherwise} \end{cases}, \quad \phi_4(x) = \begin{cases} -36(x-1/3)(x-2/3) & \text{if } 1/3 \le x \le 2/3 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_5(x) = \begin{cases} -36(x-2/3)(x-1) & \text{if } 2/3 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The derivatives are

$$\phi_3'(x) = \begin{cases} -36(2x-1/3) & \text{if } 0 \le x \le 1/3 \\ 0 & \text{otherwise} \end{cases}, \quad \phi_4'(x) = \begin{cases} -36(2x-1) & \text{if } 1/3 \le x \le 2/3 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_5'(x) = \begin{cases} -36(2x-5/3) & \text{if } 2/3 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Plots of the bubble functions are shown in Figure 7, and their derivatives in Figure 8.

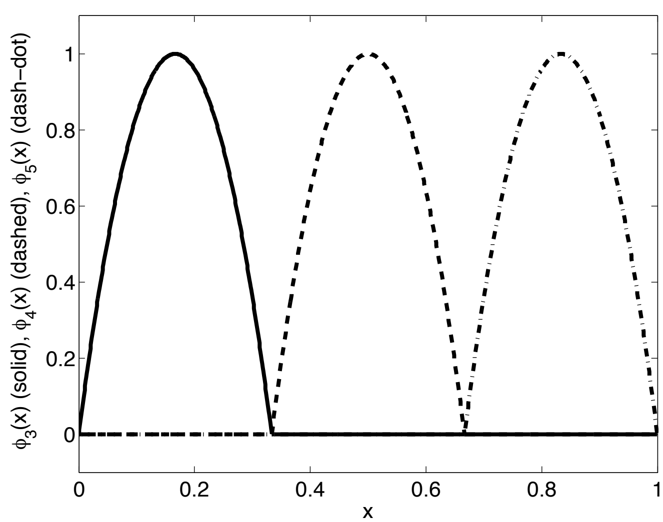

FIGURE 7. Question 3(a): plots of the bubble functions $\phi_3(x)$ (solid), $\phi_4(x)$ (dashed) and $\phi_5(x)$ (dot-dash).

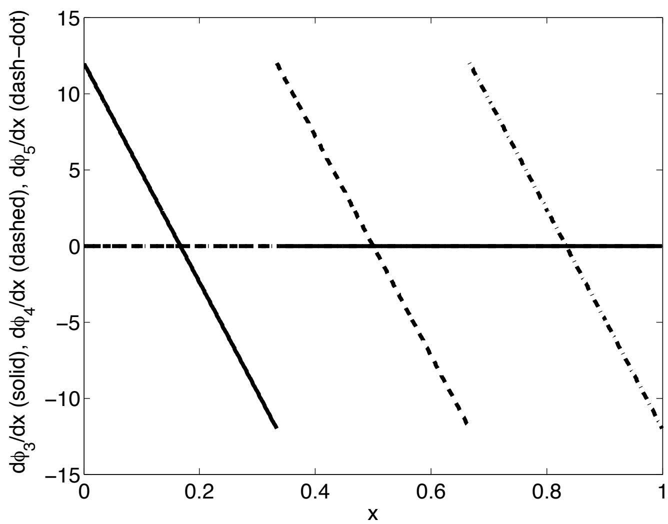

FIGURE 8. Question 3(a): plots of the derivatives of the bubble functions $\phi_3'(x)$ (solid), $\phi_4'(x)$ (dashed) and $\phi_5'(x)$ (dot-dash).

(b) We now find the matrix K. Note that K_{11}, K_{12}, K_{22} are the same as in Question 2(b). Letting $h = 1/3$, we have

$$K_{33} = \frac{16}{h^4}\int_0^h (2x-h)^2 \, \mathrm{d}x = \frac{16}{h^4}\left(\frac{h}{6}\cdot h^2 + \frac{4h}{6}\cdot 0 + \frac{h}{6}\cdot h^2\right) = \frac{16}{3h}$$

where we use Simpson's rule to evaluate the integral. (Recall that Simpson's rule is exact for quadratics.) We continue:

$$\begin{aligned}
K_{44} &= \frac{16}{h^4}\int_h^{2h} (2x-3h)^2 \, \mathrm{d}x = \frac{16}{h^4}\left(\frac{h}{6}\cdot h^2 + \frac{h}{6}\cdot h^2\right) = \frac{16}{3h} \\
K_{55} &= \frac{16}{h^4}\int_{2h}^{3h} (2x-5h)^2 \, \mathrm{d}x = \frac{6}{h^4}\left(\frac{h}{6}\cdot h^2 + \frac{h}{6}\cdot h^2\right) = \frac{16}{3h}
\end{aligned}$$

Note that $K_{15} = K_{23} = K_{34} = K_{35} = K_{45} = 0$ since the appropriate functions do not overlap (that is, their product is always zero). We continue:

$$\begin{aligned}
K_{13} &= -\frac{4}{h^3}\int_0^h (2x-h) \, \mathrm{d}x = 0 \\
K_{14} &= \frac{4}{h^3}\int_h^{2h} (2x-3h) \, \mathrm{d}x = 0
\end{aligned}$$

Both integrals follow from the symmetry of the integrand. The same is true for the rest of the matrix elements (they are all zero), so the matrix K is

$$K = \begin{pmatrix} 6 & -3 & 0 & 0 & 0 \\ -3 & 6 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 16 \end{pmatrix}$$

(c) We now compute the vector $\vec{F}$:

$$\begin{aligned} F_3 &= \int_0^1 x\phi_3(x)\,\mathrm{d}x = -\frac{4}{h^2}\int_0^h x^2(x-h)\,\mathrm{d}x = -\frac{4}{h^2}\left(\frac{h^4}{4}-\frac{h^4}{3}\right) = \frac{h^2}{3} \\ F_4 &= \int_0^1 x\phi_4(x)\,\mathrm{d}x = -\frac{4}{h^2}\int_h^{2h} x(x-h)(x-2h)\,\mathrm{d}x = -\frac{4}{h^2}\left(\frac{h^4}{4}-\frac{h^4}{2}\right) = h^2 \\ F_5 &= \int_0^1 x\phi_5(x)\,\mathrm{d}x = -\frac{4}{h^2}\int_{2h}^{3h} x(x-2h)(x-3h)\,\mathrm{d}x = -\frac{4}{h^2}\left(\frac{h^4}{4}+\frac{h^4}{3}-h^4\right) = \frac{5h^2}{3} \end{aligned}$$

The vector $\vec{F}$ thus has the form

$$\vec{F} = \begin{pmatrix} 1/9 \\ 2/9 \\ 1/27 \\ 1/9 \\ 5/27 \end{pmatrix}$$

(d) We know solve the equation $K\vec{U} = \vec{F}$, or

$$\begin{pmatrix} 6 & -3 & 0 & 0 & 0 \\ -3 & 6 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 16 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} 1/9 \\ 2/9 \\ 1/27 \\ 1/9 \\ 5/27 \end{pmatrix} \tag{0.5}$$

We found $U_1 = 4/81$ and $U_2 = 5/81$ in Problem 2(d), and the remaining are easy to find since the lower 3×3 block of K is diagonal:

$$U_3 = \frac{1}{27 \cdot 16}, \quad U_4 = \frac{1}{9 \cdot 16}, \quad U_5 = \frac{5}{27 \cdot 16}$$

The approximate solution is thus

$$U(x) = \frac{4}{81}\phi_1(x) + \frac{5}{81}\phi_2(x) + \frac{1}{27 \cdot 16}\phi_3(x) + \frac{1}{9 \cdot 16}\phi_4(x) + \frac{5}{27 \cdot 16}\phi_5(x)$$

This solution is plotted in Figure 9 (red circles). The code I used to make this plot may be found at the end of this document.

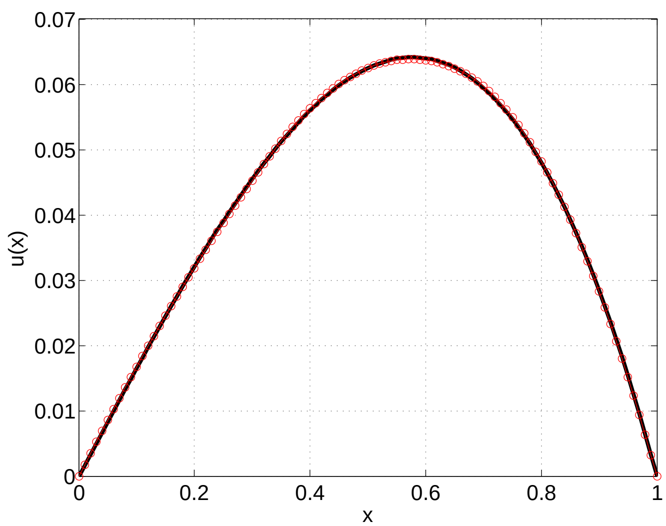

FIGURE 9. Question 3(d) and 3(e): Plot of the approximate solution (red circles) and exact solution (black line) using bubble functions.

(e) The exact solution is shown in Figure 9. Note that the inclusion of bubble functions improves the accuracy of the solution.

We find that the maximum error is at $x \approx 0.93$ and has size ≈ 0.0003, a big improvement over the error using hat functions found in Question 2(e).

Question 4. (15 pts.) A hanging beam. As shown in class, the solution for this problem has the form

$$u(x) = \begin{cases} \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D & \text{if } x < 0, \\ \frac{A+1}{6}x^3 + \frac{B}{2}x^2 + Cx + D & \text{if } x > 0 \end{cases}$$

(a) We impose the boundary conditions $u = u' = 0$ and $x = 1$ and $x = -1$. The condition $u(-1) = 0$ implies that

$$\begin{aligned} u(-1) = 0 \quad &\Rightarrow \quad -\frac{A}{6} + \frac{B}{2} - C + D = 0 \\ u(1) = 0 \quad &\Rightarrow \quad \frac{A+1}{6} + \frac{B}{2} + C + D = 0 \\ u'(-1) = 0 \quad &\Rightarrow \quad \frac{A}{2} - B + C = 0 \\ u'(1) = 0 \quad &\Rightarrow \quad \frac{A+1}{2} + B + C = 0 \end{aligned}$$

After some algebra, we find that the solution is

$$A = -1/2, \quad B = -1/4 \quad C = 0, \quad D = 1/24$$

The solution is thus

$$u(x) = \begin{cases} -\frac{x^3}{12} - \frac{x^2}{8} + \frac{1}{24} & \text{if } x \leq 0 \\ \frac{x^3}{12} - \frac{x^2}{8} + \frac{1}{24} & \text{if } x > 0 \end{cases}$$

A graph of the solution is shown in Figure 10.

(b) We now solve the problem with the boundary conditions $u = u'' = 0$ at $x = 1$ and $x = -1$.

$$\begin{aligned} u(-1) = 0 \quad &\Rightarrow \quad -\frac{A}{6} + \frac{B}{2} - C + D = 0 \\ u(1) = 0 \quad &\Rightarrow \quad \frac{A+1}{6} + \frac{B}{2} + C + D = 0 \\ u''(-1) = 0 \quad &\Rightarrow \quad -A + B = 0 \\ u''(-1) = 0 \quad &\Rightarrow \quad A + 1 + B = 0 \end{aligned}$$

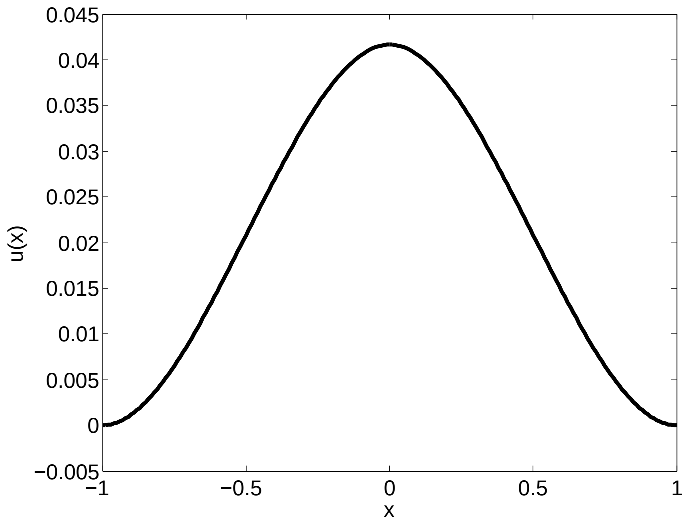

FIGURE 10. Solution to Question 4(a).

The last two equations immediately imply that $B = -1/2$ and $A = -1/2$. Substituting this into the first two equations, we find that

$$\begin{aligned} -C + D &= \frac{1}{6} \\ C + D &= \frac{1}{6} \end{aligned}$$

This implies that $C = 0$ and $D = 1/6$. The solution is thus

$$u(x) = \begin{cases} -\frac{x^3}{12} - \frac{x^2}{4} + \frac{1}{6} & \text{if } x \leq 0 \\ \frac{x^3}{12} - \frac{x^2}{4} + \frac{1}{6} & \text{if } x > 0 \end{cases}$$

A graph of the solution is shown in Figure 11.

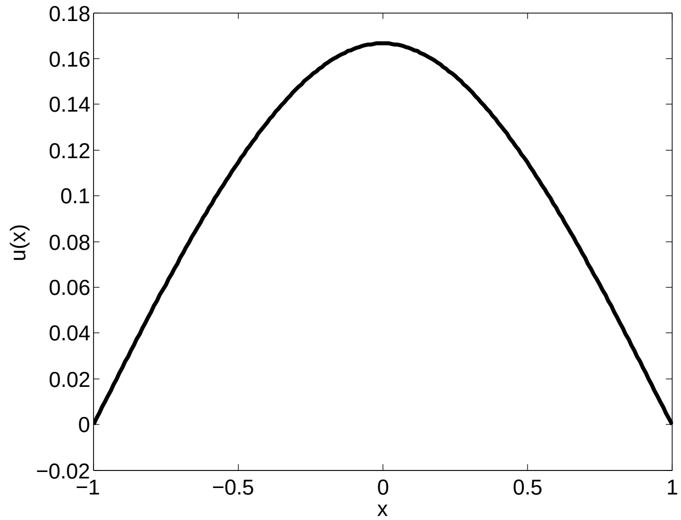

FIGURE 11. Solution to Question 4(b).

Code used to generate plots in Question 2 and 3.

```
function Pset5Q3


K = [6 -3 0 0 0; -3 6 0 0 0; 0 0 16 0 0; 0 0 0 16 0; 0 0 0 0 16];

F = [1/9 2/9 1/27 1/9 5/27]';

U = K\F;

disp(U)

disp([4/81 5/81 1/(27*16) 3/(27*16) 5/(27*16)])


xv = 0:.01:1;


usoln = U(1)*phi1(xv) + U(2)*phi2(xv) + U(3)*phi3(xv) + U(4)*phi4(xv)+U(5)*phi5(xv);

u2soln = 1/27*(4/3*phi1(xv) + 5/3*phi2(xv));

figure(20)

plot(xv,(xv-xv.^3)/6,'-k','LineWidth',3)

hold on

plot(xv,usoln,'o-r')
```

```
hold on
plot(xv,u2soln,'-r','LineWidth',3)
grid on

[a,b] = max(abs(usoln-(xv-xv.^3)/6));
disp([a xv(b)])

function out = phi1(xv)
h = 1/3;
out = zeros(1,length(xv));
for ind = 1:length(xv)
    x = xv(ind);
    if (x <= h)
        out(ind) = x/h;
    elseif (x > h && x < 2*h)
        out(ind) = 2-x/h;
    end;
end;

function out = phi2(xv)
h = 1/3;
out = zeros(1,length(xv));
for ind = 1:length(xv)
    x = xv(ind);
    if (x >= h && x < 2*h)
        out(ind) = x/h-1;
    elseif (x >= 2*h && x < 3*h)
        out(ind) = 3-x/h;
    end;
end;
```

```
function out = phi3(xv)
h = 1/3;
out = zeros(1,length(xv));
for ind = 1:length(xv)
    x  = xv(ind);
    if (x < h)
        out(ind) = -4*x/h^2*(x-h);
    end;
end;

function out = phi4(xv)
h = 1/3;
out = zeros(1,length(xv));
for ind = 1:length(xv)
    x = xv(ind);
    if (x >=h && x < 2*h)
        out(ind) = -4/h^2*(x-h)*(x-2*h);
    end;
end;

function out = phi5(xv)
h = 1/3;
out = zeros(1,length(xv));
for ind = 1:length(xv)
    x = xv(ind);
    if (x >=2*h && x < 3*h)
        out(ind) = -4/h^2*(x-2*h)*(x-3*h);
    end;
end;
```