18.085, PROBLEM SET 5, DUE 7/21 (IN CLASS)

Question 1. (15 pts.) A hanging bar.

Consider a hanging bar on the interval $0 \le x \le 1$ with a point load at $x = 1/2$, so that $f(x) = \delta(x - 1/2)$. The bar is not uniform, so $c(x) = 1/2$ for $x < 1/2$ and $c(x) = 2$ for $x > 1/2$. The equation governing the displacement $u(x)$ is

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(c(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f(x)$$

with boundary conditions $u(0) = 0$ and $w(1) = 0$, where $w(x) = c(x)u'(x)$ is the internal force on the bar due to elongation.

(a) Find $w(x)$ and graph it on the interval $[0, 1]$.

(b) Find $u'(x)$ and graph it on the same interval.

(c) Find $u(x)$ and graph it on the same interval.

Question 2. (35 pts.) Finite element method with hat functions.

In this question, we will use the finite element method to find an approximate solution to the equation

$$-u'' = x$$

with boundary conditions $u(0) = u(1) = 0$ (a uniform bar held at both ends with a load $f(x) = x$).

For this problem, the trial functions $\phi_i(x)$ will be hat functions, and the test functions $v_i(x)$ will be the same, so $v_i(x) = \phi_i(x)$. We will take $h = \Delta x = 1/3$.

(a) How many hat functions ϕ_i do you need? (Think about the boundary conditions.) Plot the hat functions together on one plot, and their derivatives on another plot.

(b) Find the matrix K by computing the integrals exactly (that is, don't use the midpoint rule.) Feel free to use the properties of the matrix K and functions ϕ_i to reduce your workload.

(c) Find the vector $\vec{F}$. This time, you can do the integrals exactly or use the midpoint rule. (In this particular example, it turns out that the midpoint rule gives the exact answer!)

(d) Solve the system $K\vec{U} = \vec{F}$ by hand (that is, you should not need Matlab.) Plot the approximate solution $U(x)$ on the interval [0,1].

(e) Find the exact solution to the equation, and plot it on top of the approximate solution you derived in part (d). Verify that the approximation $U(x)$ matches the exact solution at the nodes $x = 0, 1/3, 2/3, 1$. (You should do this by hand, not using the plot.) At what point x is the error largest, and how large is the error?

Question 3. (35 pts.) Finite element method with bubble functions.

We will now solve the same equation $-u'' = x$ with boundary conditions $u(0) = u(1) = 0$, but this time will incorporate bubble functions (parabolas).

(a) Write down the formulas for the bubble functions you will use. Plot them together on one plot, and their derivatives on another plot.

(b) Find the new matrix K. Make sure to exploit the properties of the functions to reduce your workload. Hint: Simpson's rule is exact for quadratics, and it's easier than doing all of the integrals explicitly.

(c) Find the vector $\vec{F}$. This time, you will have to do a few of the integrals explicitly, as there isn't an easy rule to compute them.

(d) Solve the system $K\vec{U} = \vec{F}$ by hand (again, you should not need Matlab.) Plot the approximate solution $U(x)$ on the interval [0,1].

(e) Plot the exact solution you found in Problem 2(e) on top of the approximate solution. Compare your answer to the approximate solution you found in Problem 2.

Question 4. (15 pts.) A hanging beam.

We will solve the equation for a uniform beam on the interval $[-1, 1]$ with a point load at $x = 0$:

$$u'''' = \delta(x)$$

(a) Solve the equation for a built-in-beam, with boundary conditions $u = u' = 0$ at the ends $x = -1$ and $x = 1$. Graph your solution.

(b) Solve the equation for a simply supported beam, with $u = u'' = 0$ at the ends $x = -1$ and $x = 1$. Graph your solution.