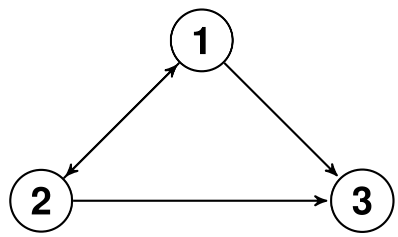
18.085: PROBLEM SET 4 SOLUTIONS

Question 1. (30 pts.) PageRank.

(a) The appropriate graph looks like:



(b) The modified adjacency matrix is

$$L = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

For instance, the first column corresponds to node 1. It links to nodes 2 and 3, so each gets a weight of 1/2, hence the 1/2 in the second and third entries of the first column. The same logic may be applied for the other columns.

(c) We want to show that one of the eigenvalues of L is 1. The characteristic polynomial of L is

$$p(\lambda) = \det(L - \lambda I) = \begin{vmatrix} -\lambda & 1/2 & 0 \\ 1/2 & -\lambda & 1 \\ 1/2 & 1/2 & -\lambda \end{vmatrix} = -\lambda\left(\lambda^2 - \frac{1}{2}\right) - \frac{1}{2}\left(-\frac{\lambda}{2} - \frac{1}{2}\right)$$

It is clear that $p(1) = -\frac{1}{2} - \frac{1}{2} \cdot (-1) = 0$, so $\lambda = 1$ is a root of $p(\lambda)$ and hence an eigenvalue of L. The corresponding eigenvector satisfies the equation $L\mathbf{R} = \mathbf{R}$, or in component form

$$\frac{R_2}{2} = R_1, \quad \frac{R_1}{2} + R_3 = R_2, \quad \frac{R_1}{2} + \frac{R_2}{2} = R_3$$

Substituting the first equation into the second (or the third), we obtain $R_3 = 3R_2/4$. We thus take $R_1 = 1, R_2 = 2, R_3 = 3/2$. The sum of the components is 9/2, so we divide each component by 9/2 to obtain a normalized vector (that is, a vector whose components sum to 1):

$$\mathbf{R} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} 2/9 \\ 4/9 \\ 1/3 \end{pmatrix}$$

Webpage 1 has the lowest score of 2/9, followed by webpage 3 (1/3) and webpage 2 (4/9).

(d) A sample code to execute the power iteration is given below:

```
v = rand(3,1);
L = [0 1/2 0; 1/2 0 1; 1/2 1/2 0];
for ind = 1:20
    v = L*v/norm(L*v);
    disp(v/sum(v));
end;
```

After 20 iterations, the vector $\mathbf{v}$ approximates the eigenvector $\mathbf{R} = (2/9, 4/9, 1/3)$ from part (c) to four decimal places.

Question 2. (40 pts.) Trusses.

(a) The matrix A corresponding to the truss is

$$A = \begin{pmatrix} \cos\tilde{\theta}_1 & \sin\tilde{\theta}_1 & -\cos\tilde{\theta}_1 & -\sin\tilde{\theta}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos(\pi-\theta_2) & \sin(\pi-\theta_2) & 0 & 0 & -\cos(\pi-\theta_2) & -\sin(\pi-\theta_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

where $\tilde{\theta}_1 = \pi - \theta_1 - \theta_2$ is the angle between edges 1 and 3.

We can simplify this matrix using the identities $\cos(\pi-\theta) = -\cos\theta$ and $\sin(\pi-\theta) = \sin\theta$.

(b) A rigid motion of the truss by a distance d_x in the x-direction and d_y in the y-direction corresponds to the vector

$$\vec{u} = \begin{pmatrix} d_x & d_y & d_x & d_y & d_x & d_y & d_x & d_y & d_x & d_y \end{pmatrix}^T$$

It is straightforward to verify that $A\vec{u} = \vec{0}$. That is, the dot product of each row of A with $\vec{u}$ vanishes.

(c) Now nodes 4 and 5 are supported, so $u_4^H = u_4^V = u_5^H = u_5^V = 0$. We remove the last four columns (two for each node) of A:

$$A_{\text{red}} = \begin{pmatrix} \cos\tilde{\theta}_1 & \sin\tilde{\theta}_1 & -\cos\tilde{\theta}_1 & -\sin\tilde{\theta}_1 & 0 & 0 \\ -\cos\theta_2 & \sin\theta_2 & 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This truss has a mechanism corresponding to rigid translation in the x-direction. That is,

$$\vec{u} = \begin{pmatrix} d & 0 & d & 0 & d & 0 \end{pmatrix}^T$$

It is straightforward to verify that $A_{\text{red}}\vec{u} = \vec{0}$, so the truss is unstable.

(d) We return to the *unsupported* truss for the time being and add a bar (#6) between nodes 4 and 3. Let ϕ be the angle that edge 6 makes with the horizontal. We add one row to the matrix A defined in part (a):

$$A = \begin{pmatrix} \cos\tilde{\theta}_1 & \sin\tilde{\theta}_1 & -\cos\tilde{\theta}_1 & -\sin\tilde{\theta}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos(\pi-\theta_2) & \sin(\pi-\theta_2) & 0 & 0 & -\cos(\pi-\theta_2) & -\sin(\pi-\theta_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cos\phi & \sin\phi & -\cos\phi & -\sin\phi & 0 & 0 \end{pmatrix}$$

We now support nodes 4 and 5, and so remove the last 4 columns from A:

$$A_{\text{red}} = \begin{pmatrix} \cos\tilde{\theta}_1 & \sin\tilde{\theta}_1 & -\cos\tilde{\theta}_1 & -\sin\tilde{\theta}_1 & 0 & 0 \\ -\cos\theta_2 & \sin\theta_2 & 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cos\phi & \sin\phi \end{pmatrix}$$

To show that the truss is stable, we want to show that $A_{\text{red}}\vec{u} = \vec{0}$ has no solutions (other than the obvious one $\vec{u} = \vec{0}$). In other words, we want to show that the columns of A_{red} are independent, or that $\det(A_{\text{red}}) \neq 0$. Finding the determinant is not that hard, since the matrix is mostly zeros. Indeed, we can expand along the first column to obtain

$$\begin{aligned} \det(A_{\text{red}}) &= \cos\tilde{\theta}_1 \left(\sin\theta_2 \cdot (-1) \cdot 1 \cdot (-\cos\phi)\right) + \cos\theta_2 \left(\sin\tilde{\theta}_1 \cdot (-1) \cdot 1 \cdot (-\cos\phi)\right) \\ &= \cos\phi \left(\cos\tilde{\theta}_1 \sin\theta_2 + \cos\theta_2 \sin\tilde{\theta}_1\right) \\ &= \cos\phi \sin(\tilde{\theta}_1 + \theta_2) \end{aligned}$$

Note that this is nonzero as long as $\phi \neq \pi/2$ and $\tilde{\theta}_1 + \theta_2 \neq \pi$, both of which are satisfied for the truss.

(e) Adding a bar between nodes 4 and 5 does not make the truss stable. The full matrix A, with nodes 4 and 5 *unsupported* will look like the one in part (a), except with the additional row

$$r_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

When nodes 4 and 5 are supported, you lose the last four columns of A, so the last row of A_{red} is all zeros:

$$A_{\text{red}} = \begin{pmatrix} \cos\tilde{\theta}_1 & \sin\tilde{\theta}_1 & -\cos\tilde{\theta}_1 & -\sin\tilde{\theta}_1 & 0 & 0 \\ -\cos\theta_2 & \sin\theta_2 & 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Despite the fact that A_{red} is 6×6, it is clear that it does not have full rank. That is, its columns (and rows) are not independent, and its determinant is zero. This follows from the fact that the last row is all zeros.

(f) Placing node 5 on a roller means that $u_5^V = 0$ but $u_5^H \neq 0$; that is, the node can move horizontally but not vertically. We thus remove columns 7, 8 and 10 from the original matrix A in part (a), but keep column 9 (which corresponds to u_5^H):

$$A_{\text{red}} = \begin{pmatrix} \cos\tilde{\theta}_1 & \sin\tilde{\theta}_1 & -\cos\tilde{\theta}_1 & -\sin\tilde{\theta}_1 & 0 & 0 & 0 \\ -\cos\theta_2 & \sin\theta_2 & 0 & 0 & \cos\theta_2 & -\sin\theta_2 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The matrix is 5×7, so we expect $A_{\text{red}}\vec{u} = \vec{0}$ to have at least two solutions. They are

$$\begin{aligned} \vec{u}_1 &= \begin{pmatrix} d & 0 & d & 0 & d & 0 & d \end{pmatrix}^T \\ \vec{u}_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & d \end{pmatrix} \end{aligned}$$

The first corresponds to rigid motion in the x-direction, and the second corresponds to moving node 5 alone in the x-direction. You can check that $A_{\text{red}}\vec{u} = \vec{0}$ for both of these vectors. The truss is thus unstable.

Question 3. (30 pts.) Newton's method.

(a) The first equation implies that $y = x^3$. Substituting into the second equation, we obtain $x^9 = x$, or $x(x^8 - 1) = 0$. This implies that $x = 0$, $x = 1$, or $x = -1$. The corresponding values for y are $y = 0$, $y = 1$ and $y = -1$. So the solutions are $(x, y) = (0, 0)$, $(1, 1)$ and $(-1, -1)$.

(b) Let $f(x, y) = x^3 - y$ and $g(x, y) = y^3 - x$. The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2 & -1 \\ -1 & 3y^2 \end{pmatrix}$$

(c) Newton's method is defined by the iteration

$$J(\mathbf{u}_n)(\mathbf{u}_{n+1} - \mathbf{u}_n) = -\mathbf{F}(\mathbf{u}_n) \Rightarrow \mathbf{u}_{n+1} = \mathbf{u}_n - J(\mathbf{u}_n)^{-1}\mathbf{F}(\mathbf{u}_n)$$

where the Jacobian matrix J was defined in part (b), $\mathbf{u} = (x, y)$ and $\mathbf{F} = (f(x, y), g(x, y)) = (x^3 - y, y^3 - x)$. Starting with $(x, y) = (2, 1)$, the first 10 iterates of Newton's method are

$$\mathbf{u}_1 = (2, 1), \quad \mathbf{u}_2 = (1.4286, 1.1429), \quad \mathbf{u}_3 = (1.1237, 1.0487), \quad \mathbf{u}_4 = (1.0149, 1.0067),$$
$$\mathbf{u}_5 = (1.0003, 1.0001), \quad \mathbf{u}_6 \text{ through } \mathbf{u}_{10} = (1, 1) \text{ to four decimal places.}$$

The code to generate these values is given below:

```
u = [2;1];
for ind = 1:10
    x = u(1); y = u(2);
    J = [3*x^2 -1; -1 3*y^2];
    F = [x^3-y; y^3-x];
    u = u - J\F; disp(u);
end;
```

(d) See Figure 1. Note that an initial guess does not always converge to the closest solution, and many initial guesses (indicated in black) do not converge to a solution at all. The code used to generate Figure 1 is given below:

```
function PSet4Q3

N = 50;
tol = 1e-5;

r1 = [0;0];
r2 = [1;1];
r3 = [-1;-1];

xv = linspace(-5,5,N);
yv = linspace(-5,5,N);

nT = 10;

for ix = 1:length(xv)
    for iy = 1:length(yv)
        x = xv(ix);
        y = yv(iy);
        z = [x;y];
        for j = 1:nT
            f = [z(1)^3-z(2) z(2)^3-z(1)]';
            J = [3*z(1)^2 -1; -1 3*z(2)^2];
            z = z - J\f;
        end;
        if norm(z-r1) < tol
            c = 'r';
        elseif norm(z-r2) < tol
            c = 'b';
        elseif norm(z-r3) < tol
            c = 'g';
```

```
            else
                c = 'k';
            end;
            figure(1)
            plot(x,y,'.','color',c,'MarkerSize',20);
            hold on
        end;
    end;
```

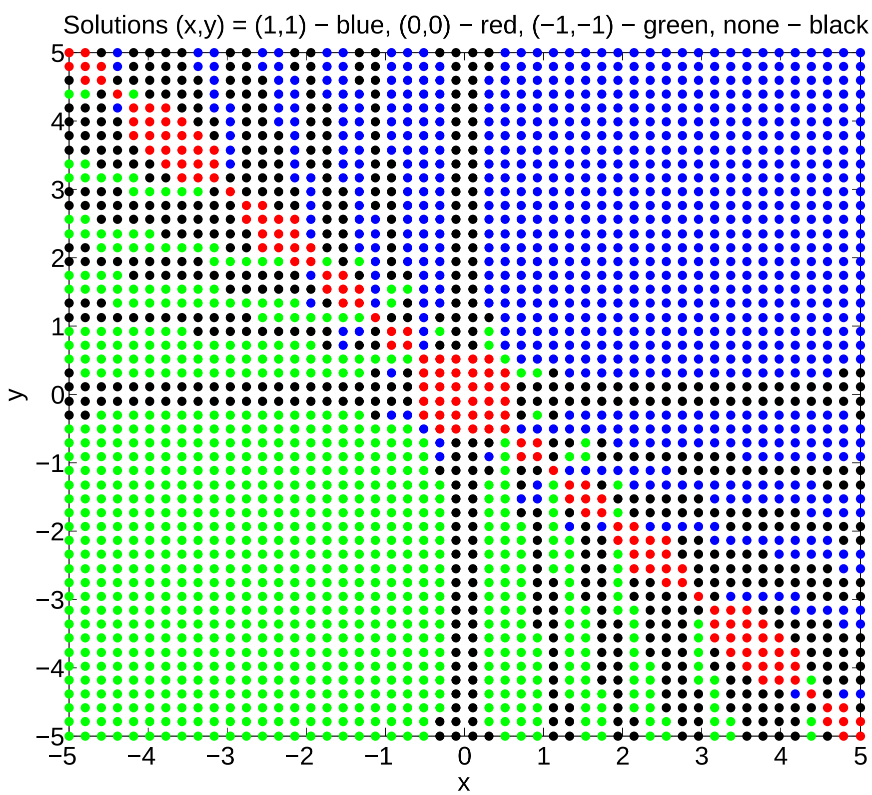


FIGURE 1. Solution Question 3

Bonus (10 pts.) Newton's method with a vectorized code.

See Figure 2. In order to vectorize the code, we cannot use backslash or `inv(J)*F`, since X and Y are both matrices. That is, each element of J and F is itself a matrix! To get around this, we need to write Newton's method in component form. We need to use the formula for the inverse of J given in the hint. That is,

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \frac{1}{j_1 j_4 - j_2 j_3} \begin{pmatrix} j_4 & -j_3 \\ -j_2 & j_1 \end{pmatrix} \begin{pmatrix} x_n^3 - y_n \\ y_n^3 - x_n \end{pmatrix}$$

where j_i are the matrix elements of the Jacobian you found in part (b):

$$J = \begin{pmatrix} 3x^2 & -1 \\ -1 & 3y^2 \end{pmatrix} \Rightarrow j_1 = 3x^2, \quad j_2 = -1, \quad j_3 = -1, \quad j_4 = 3y^2$$

In other words, we have

$$\begin{aligned} x_{n+1} &= x_n - \frac{1}{9x_n^2 y_n^2 - 1} \left(3y_n^2(x_n^3 - y_n) + 1 \cdot (y_n^3 - x_n)\right) \\ y_{n+1} &= y_n - \frac{1}{9x_n^2 y_n^2 - 1} \left(1 \cdot (x_n^3 - y_n) + 3x_n^2(y_n^3 - x_n)\right) \end{aligned}$$

This is implemented in the code below, which was used to generate Figure 2. You can see the basins of attraction quite clearly in this figure.

```
function PSet4Q3_vect


N = 1000;
tol = 1e-5;


xv = linspace(-5,5,N);
yv = linspace(-5,5,N);


[X,Y] = meshgrid(xv,yv);
X0 = X;
Y0 = Y;


nT = 10;


for j = 1:nT
```

```
    f1 = X.^3-Y;
    f2 = Y.^3-X;
    J1 = 3*(X.^2);
    J2 = -1*ones(N,N);
    J3 = J2;
    J4 = 3*(Y.^2);
    dJ = J1.*J4-J2.*J3;
    Ji1 = J4./dJ;
    Ji2 = -J3./dJ;
    Ji3 = -J2./dJ;
    Ji4 = J1./dJ;
    X = X - (Ji1.*f1 + Ji2.*f2);
    Y = Y - (Ji3.*f1 + Ji4.*f2);
end;
Nrm1 = sqrt(X.^2+Y.^2);
Nrm2 = sqrt((X-1).^2+(Y-1).^2);
Nrm3 = sqrt((X+1).^2+(Y+1).^2);
C = zeros(N,N);
C(Nrm1 < tol) = 1;
C(Nrm2 < tol) = 2;
C(Nrm3 < tol) = 3;
figure(3)
surf(X0,Y0,rand(N,N),C,'edgecolor','none'); view([0,90])
```

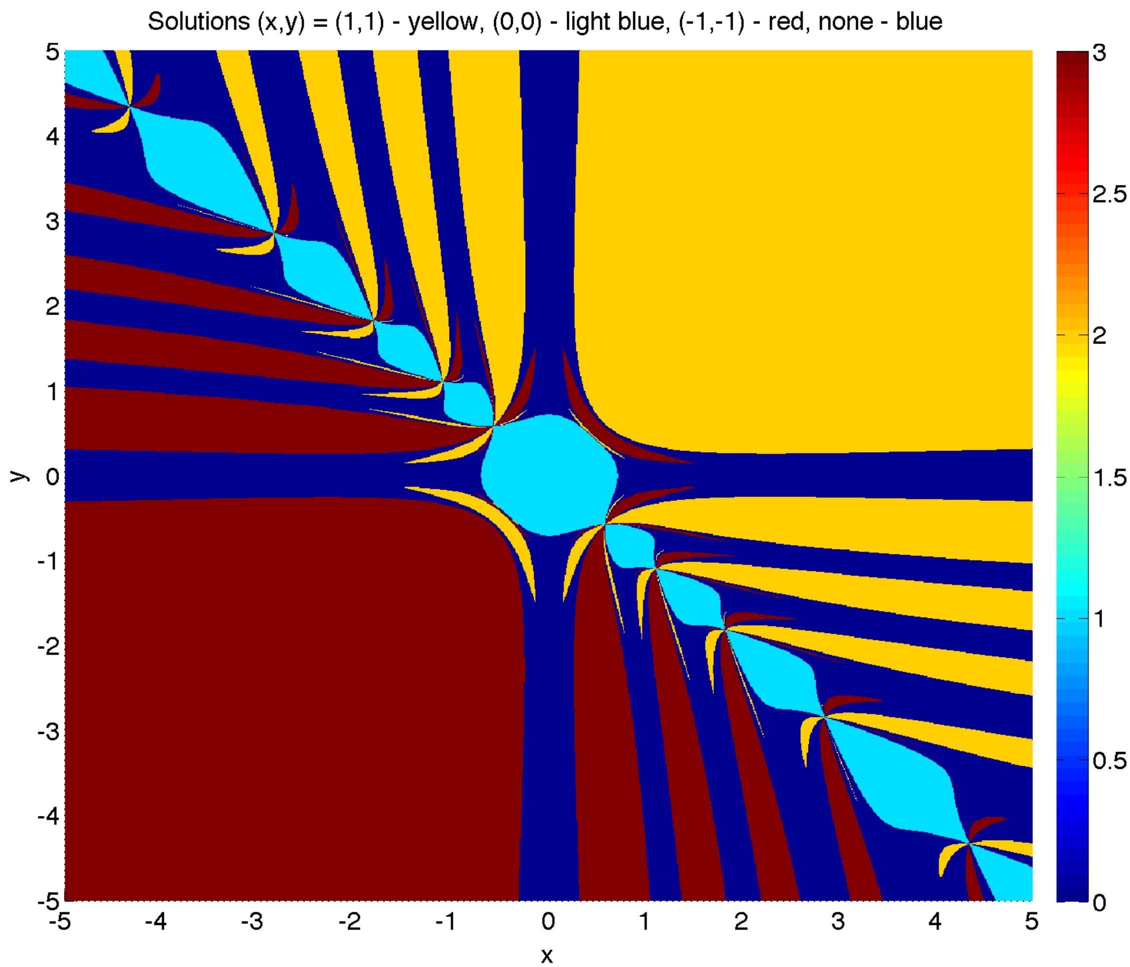


Figure 2. Solution to bonus question.