18.085, PROBLEM SET 2 SOLUTIONS

Question 1. (25 pts.) Delta function practice.
Please solve the following equation, with 2 point loads and free-fixed boundary conditions:

$$-u''(x) = \delta(x - 1/2) + \delta(x - 2/3), \ u'(0) = 0, \ u(1) = 0.$$

a) Here is how to find the solution. First, we know we expect the solution to have $-R(x-1/2)$ as the particular solution corresponding to the delta function $\delta(x-1/2)$, and to have $-R(x-2/3)$ as the particular solution corresponding to the delta function $\delta(x-2/3)$, and finally $Cx + D$, the homogeneous solution. Hence

$$u(x) = -R(x - 1/2) - R(x - 2/3) + Cx + D,$$

and we determine the constants C and D using the boundary conditions.

$$u'(0) = -0 - 0 + C = 0 \Rightarrow C = 0$$

(since the slope of $-R(0-a)$ is 0, the ramps have not started yet) and

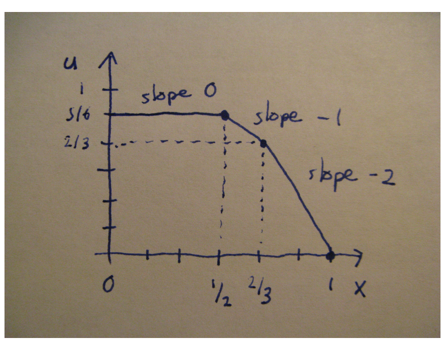
$$u(1) = -R(1 - 1/2) - R(1 - 2/3) + D = 0 \Rightarrow -(1 - 1/2) - (1 - 2/3) + D = 0 \Rightarrow D = 5/6,$$

since the ramps are linear of slope 1, the value of $R(x-a)$ at $x > a$ is $R(x-a) = (x-a)$. So we have the solution, written both ways:

$$u(x) = -R(x - 1/2) - R(x - 2/3) + 5/6,$$

$$u(x) = \begin{cases} 5/6 & 0 \le x \le 1/2 & \text{(ramps haven't kicked in yet)} \\ 4/3 - x & 1/2 \le x \le 2/3 & \text{(first ramp kicks in)} \\ 2 - 2x & 2/3 \le x \le 1 & \text{(second ramp kicks in)} \end{cases}$$

b) Here is a plot of the solution u by hand. The slope of u decreases by 1 both at $x = 1/2$ and $x = 2/3$. This is caused by each delta function.



c) Here is the new file "`pset2_1.m`", on the next page. Appropriate values of n are such that $n + 1$ is a multiple of both 2 and 3, hence a multiple of 6. Because our solution is made of straight lines, and we know finite differences are exact on constants, lines and quadratics, we know we should obtain an error of 0 (or on the order of 10^{-16}). This is indeed what you get, for various appropriate values of n such as 5, 11, 17, etc.

```
% pset2_1_sol.m
% Please replace question marks by appropriate code
%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Setup %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n=1*6-1; % appropriate value of n so delta functions fall on mesh points
h=1/(n+1); % step size
x=0:h:1-h;x=x'; % mesh points
xdelta=[1/2 2/3]; % contains points at which there is a delta function

%%%%%%%%%%%%%%%%%%%%%%%%%%% Finite differences %%%%%%%%%%%%%%%%%%%%%%%%%%%
T=sparse([],[],[],n+1,n+1,3*(n+1));
T=spdiags(ones(n+1,3)*diag([-1 2 -1]),[ -1 0 1 ],T);T(1,1)=1;
f=zeros(n+1,1); f(round(xdelta/h)+1)=(1/h);% right-hand side, ONE MORE UNKNOWN
f(1)=f(1)/2; % fix right-hand side for second order accuracy at boundary


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Solution %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
u=(1/h^2)*T\f;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Error %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The following creates the true solution, piece-wise
% The code (a<=x)&(x<b) creates a vector the size of x, with 0 everywhere
% except where BOTH a<=x AND x<b.
ut=(5/6  ).*((  0<=x)&(x<1/2))+...
   (8/6-x).*((1/2<=x)&(x<2/3))+...
   (2-2*x).*((2/3<=x)&(x<1  ));
e=sqrt(h*sum(abs(u-ut).^2));disp(e); % L2 error in interval [0,1]
```

Question 2. (30 pts.) Finite differences, full-fledged.

a) The differential equation is

$$-u'' = \delta(x-1/2) + \delta(x-2/3) + \cos(x\pi/2), \qquad u'(0)=0, \qquad u(1)=0.$$

b) The true solution is $u(x) = -R(x-1/2) - R(x-2/3) + 5/6 + (4\pi^2)\cos(x\pi/2)$.

c) Here is the new file "`pset2_2.m`", on the next page.

d) We try a few different appropriate values of n $(5, 11, 23, 47, 95)$, and obtain the loglog plot on figure , showing a slope of -2. This is what we expect from our method having second order accuracy, that is, the error should behave like $O(h^2)$.

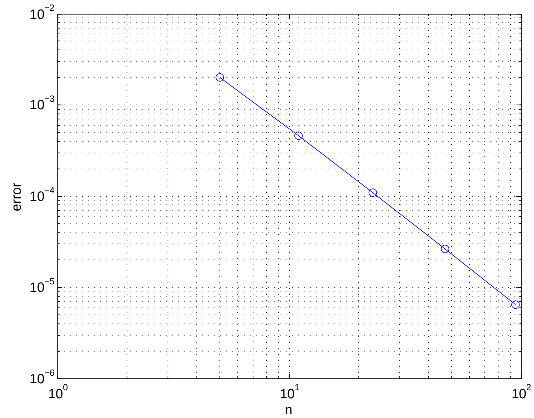


FIGURE 1. Error in Q2.

```
% pset2_2_sol.m

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Setup %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n=4*6-1; % appropriate value of n so delta functions fall on mesh points
h=1/(n+1); % step size
x=0:h:1-h;x=x'; % mesh points
xdelta=[1/2 2/3]; % contains points at which there is a delta function

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Finite differences %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
T=sparse([],[],[],n+1,n+1,3*(n+1));
T=spdiags(ones(n+1,3)*diag([-1 2 -1]),[ -1 0 1 ],T);T(1,1)=1;
f=zeros(n+1,1); f(round(xdelta/h)+1)=(1/h);
f=f+cos(pi*x/2);% right-hand side
f(1)=f(1)/2; % fix right-hand side for second order accuracy at boundary

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Solution %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
u=(1/h^2)*T\f;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Error %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The following creates the true solution, piece-wise
% The code (a<=x)&(x<b) creates a vector the size of x, with 0 everywhere
% except where BOTH a<=x AND x<b.
ut=(5/6  ).*((  0<=x)&(x<1/2))+...
   (8/6-x).*((1/2<=x)&(x<2/3))+...
   (2-2*x).*((2/3<=x)&(x<1  ));
ut=ut+(4/pi^2)*cos(pi*x/2);
%e=sqrt(h*sum(abs(u-ut).^2));disp(e); % L2 error in interval [0,1]
e=abs(u(round(1/2/h))-ut(round(1/2/h)))
```

Question 3. (10 pts.) Delta function in the periodic case.

Again, we expect the solution to have a particular and a homogeneous part:

$$u(x) = -R(x-a) + Cx + D,$$

and we (try to) find the constants C and D using $u(0) = u(1)$ and $u'(0) = u'(1)$. Going straight to the second condition, we have $u'(0) = -0+C$ and $u'(1) = -1+C$. This is because the derivative of the ramp is 0 at $x = 0$ (ramp hasn't started) but it is 1 at $x = 1$ because now the ramp has started. However, we need $u'(0) = u'(1)$, which is clearly impossible. Hence we cannot find a solution to this problem. Thism eans that, if we discretized this system, we would not be able to solve it: $C_n\vec{u} = \vec{\delta}$.

Question 4. (10 pts.) Orthonormal eigenvectors.

We saw in class that matrix K is symmetric and thus has a full set of orthogonal eigenvectors. We like to have orthogonal eigenvectors because of the following, which we show for K_2 but holds in general.

a) We know the eigenvalues of K_2 are $\lambda_1 = 1$ and $\lambda_2 = 3$. The eigenvectors are $\vec{y_1} = (1\ 1)^T$ and $\vec{y_2} = (1\ -1)^T$, and we make them unit vectors: $\vec{q_1} = (1\ 1)^T/\sqrt{2}$ and $\vec{q_2} = (1\ -1)^T/\sqrt{2}$. Thus we indeed have that $K_2\vec{q_j} = \lambda_j\vec{q_j}$ and $\vec{q_j}^T\vec{q_j} = 1$ for $j = 1, 2$.

b) Verify that the eigenvectors $\vec{q_1}$ and $\vec{q_2}$ are orthonormal (orthogonal and of unit length): we already know they have norm 1, and they are orthogonal since $\vec{y_1}^T\vec{y_2} = (1\times 1 - 1\times 1)/2 = 0$.

c) Verify by hand that $K_2 = Q\Lambda Q^T$ where Λ is the matrix with the eigenvalues on its diagonal, and Q is the matrix with columns $\vec{q_1}$ and $\vec{q_2}$:

$$Q\Lambda Q^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \times \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \times \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 3/\sqrt{2} & -3/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 4/2 & -2/2 \\ -2/2 & 4/2 \end{pmatrix} = K_2$$

This is not a coincidence! When a matrix has orthogonal eigenvectors, as do symmetric matrices, we can decompose it as a product of the matrix of orthonormalized eigenvectors multiplied by the matrix of eigenvalues, multiplied by the *transpose* (and not the *inverse* as usual) of the matrix of orthonormalized eigenvectors.

d) Verify by hand that $Q^TQ = I$ and $QQ^T = I$ as well, where I is the 2 by 2 identity matrix. This confirms that $Q^T = Q^{-1}$! It happens here that in fact $Q^T = Q$, although that might depend on the y_2 you choose in a). Then we have

$$QQ^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \times \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2/2 & 0 \\ 0 & 2/2 \end{pmatrix} = I,$$

and similarly for Q^TQ.

When a matrix has orthonormal columns, it's inverse is very easy to take: jsut take the transpose! And when a matrix has orthonormal eigenvectors, its inverse is also very easy: $K_2^{-1} = (Q\Lambda Q^T)^{-1} = (Q^T)^{-1}\Lambda^{-1}Q^{-1} = Q\Lambda^{-1}Q^T$!

Question 5. (10 pts.) From eigenvectors to eigenfunctions.

We saw in pset1 Q2 that $-\frac{d^2}{dx^2}\sin(\pi x) = \pi^2 \sin(\pi x)$ with fixed-fixed boundary conditions... It looks as if π^2 is an "eigenvalue" of the operator $-\frac{d^2}{dx^2}$, and that $\sin(\pi x)$ is the corresponding "eigenvector". More generally, we have $-\frac{d^2}{dx^2}\sin(k\pi x) = k^2\pi^2\sin(k\pi x)$ for non-zero integer k.

a) We have $n = 6$ and $h = 1/7$, and in Matlab "`K=toeplitz([2 -1 0 0 0 0])`" and "`[Q E]=eig((1/h^2)*K);`". This gives you the eigenvectors as columns of Q, and eigenvalues in the diagonal of E. We see that the diagonal values of E are the same as "`(2-2*cos(k*pi*h))/h^2`", for "`k=[1:6];`":

```
Q =
   2.3192e-01  -4.1791e-01  -5.2112e-01   5.2112e-01   4.1791e-01   2.3192e-01
   4.1791e-01  -5.2112e-01  -2.3192e-01  -2.3192e-01  -5.2112e-01  -4.1791e-01
   5.2112e-01  -2.3192e-01   4.1791e-01  -4.1791e-01   2.3192e-01   5.2112e-01
   5.2112e-01   2.3192e-01   4.1791e-01   4.1791e-01   2.3192e-01  -5.2112e-01
   4.1791e-01   5.2112e-01  -2.3192e-01   2.3192e-01  -5.2112e-01   4.1791e-01
   2.3192e-01   4.1791e-01  -5.2112e-01  -5.2112e-01   4.1791e-01  -2.3192e-01
>> diag(E)'
   9.7051e+00   3.6898e+01   7.6193e+01   1.1981e+02   1.5910e+02   1.8629e+02
>> (2-2*cos([1:6]*pi*h))/h^2
   9.7051e+00   3.6898e+01   7.6193e+01   1.1981e+02   1.5910e+02   1.8629e+02
```

b) Show that, in the limit as $h \to 0$, we recover the eigenvalues we expect from the discussion above, that is:

$$\lim_{h\to} \frac{1}{h^2}(2 - 2\cos(k\pi h)) = k^2\pi^2.$$

From the hint, we know $cos(k\pi h) \approx 1 - (k\pi h)^2/2$ for small h. Hence

$$\lim_{h\to} \frac{1}{h^2}(2 - 2\cos(k\pi h)) = \lim_{h\to} \frac{1}{h^2}(2 - 2(1 - (k\pi h)^2/2)) = k^2\pi^2,$$

as required. In fact those are the eigenvalues we expect for K_n, any n. A course in numerical analysis would carefully prove this, but this is not the point of 18.085.

c) We would expect sines to appear in the eigenvectors of K_6, again because of the discussion above. In particular, we expect the functions $\sin(k\pi x)$, discretized to $\sin(k\pi jh)$ for $j = 1, 2, \ldots, n$. Those give us the *discrete sine transform*:

"`DST=sin(JK*pi*h)*sqrt(2)/sqrt(n+1)`",

where matrix "`JK=[1:6]'*[1:6]`" has entries j times k. Indeed the eigenvector matrix Q correspond to the DST, up to a sign changes. That is, eigenvectors 2 and 3 in Q (columns 2 and 3 of Q) need to be multiplied by a minus sign to agree with the DST. Otherwise the 2 are equal.

d) Since K_6 is symmetric, we expect orthogonal eigenvectors. You can verify in Matlab that $Q^{-1} = Q^T$, by comparing say "`inv(Q)`" to "`Q'`". Same for the DST. This will be useful when we talk about the Fourier transform, and its two special cases, the discrete sine tansform and the discrete cosine transform. To invert any of those transforms, we simply need to take their transpose!

Question 6. (15 pts.) Linear algebra and differential equations practice.

a) If a matrix A has a zero eigenvalue, then there exists a non-zero vector $\vec{y}$ such that $A\vec{y} = 0\vec{y} = \vec{0}$. This means the matrix A is not invertible, because there is no way to invert the map back from the 0 vector to $\vec{y}$. (Or: the matrix does not have full rank, it has determinant 0, it does not have independent columns, etc.)

b) The matrix B (free-free boundary conditions), which is K but with top left and bottom right entries of 1, has an eigenvector with 0 eigenvalue, which is the all-ones vector. Looking at the general B_n, for any n, we can easily see that the rows sum to 0, and so the all-ones vector has to be an eigenvector with eigenvalue 0. Because this is a non-zero vector, this means B is not invertible from a), as we had discussed in class.

c) Let A be an n by n matrix with n independent eigenvectors. Let S be its eigenvector matrix. That is, the n columns of S are the eigenvectors of A, and we may write $A = S\lambda S^{-1}$, with eigenvalues of A in the diagonal of Λ. The eigenvalues of $A - 3I$ are the original eigenvalues, minus 3. Indeed, if $AS = \Lambda S$, then $(A - 3I)S = AS - 3S = \Lambda S - 3IS = (\Lambda - 3I)S$. This also shows that the eigenvectors are the same, that is, the columns of S.

d) Starting from $det(A - \lambda I) = 0$, we find the eigenvalues first: $det(A - \lambda I) = (4 - \lambda)(1 - \lambda) - 0 = 0$ implies eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$. We then find the eigenvectors. We can do this by trial and error, or find vectors that send the matrices $A - \lambda I$ to the zero vector:

$$(A - \lambda_1 I)\vec{y} = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \vec{y_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{y_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$(A - \lambda_2 I)\vec{y_2} = \begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix} \vec{y_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{y_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(You can check your answer by multiplying A by $\vec{y_1}$ and $\vec{y_2}$.)

Use those eigenvalues and eigenvectors to produce solutions $\vec{u}(t) = e^{\lambda_j t}\vec{y_j}$ to $d\vec{u}/dt = A\vec{u}$. Thus we have $\vec{u_1}(t) = e^{\lambda_1 t}\vec{y_1} = e^t\vec{y_1}$ and $\vec{u_2}(t) = e^{\lambda_2 t}\vec{y_2} = e^{4t}\vec{y_2}$. We show those are indeed solutions by plugging them in the equation $d\vec{u}/dt = A\vec{u}$. Indeed, $d\vec{u_1}/dt = e^t\vec{y_1}$ is equal to $A\vec{u_1} = e^t A\vec{y_1} = e^t \times 1 \times \vec{y_1}$. Same for $\vec{u}_2$.