18.085, PROBLEM SET 2, DUE 6/20 NOON IN E17-301Q

Question 1. (25 pts.) Delta function practice.
Please solve the following equation, with 2 point loads and free-fixed boundary conditions:

$$-u''(x) = \delta(x - 1/2) + \delta(x - 2/3), \; u'(0) = 0, \; u(1) = 0.$$

a) Write out the solution both ways: 1) in terms of ramps and other simple functions, and 2) as a piecewise function.
b) Plot the solution u by hand. What happens to the slope of u at $x = 1/2$? at $x = 2/3$?
c) Use file "`pset2_1.m`" to verify your answer numerically. Submit your new version of this file, or at least the modified lines.

There are a few things to notice in "`pset2_1.m`". First, we will need some mesh points to fall exactly on the delta functions. Think about what values of n will work for this.

Second, we are now using Matlab's sparse matrices capabilities. Have a look at lines 12 and 13 to understand what they do. I also suggest you print out this new, sparse T in Matlab's command line, to see what it looks like. For small values of n, you can try also typing "`full(T)`".

Third, please fix the code (line 14) that places the delta functions in vector f.

Then, you will need to input the true solution you found above (lines 22-27).

Finally, you will remark we find the error a different way, by integrating the square of the error over the interval $[0, 1]$. This is called the "L_2 norm", and is a better (more global) measure of how good our approximation is.

By the way, if you need Matlab help, type doc in the command line and use the search feature, or type help followed by the function you want to know about.

Question 2. (30 pts.) Finite differences, full-fledged.

We will now model an elastic bar of length 1, free at the top ($x = 0$) and fixed at the bottom ($x = 1$). The bar is almost weightless, but it does have two point loads weighing on it, at $x = 1/2$ and $x = 2/3$. It also has an other load, distributed over the whole bar, that is a cosine, more precisely $\cos(\pi x/2)$.

a) Write down the corresponding differential equation.
b) What is the true solution to this differential equation? (Hint: use your solution from pset2 Q1 of course, and have another look at pset1 Q2 too.)
c) Solve your differential equation from a) using finite differences, modifying file "`pset2_1.m`" into a new file "`pset2_2.m`" as needed. Print out this new file and submit it with your pset solution.
d) Try a few different values of n, and describe what happens to the error e. Do the same kind of plot as in the solutions to pset1, Q2. This part was optional for pset1, but it is required now.

Question 3. (10 pts.) Delta function in the periodic case.

(From the book.) Show that $-u'' = \delta(x - a)$ with *periodic* conditions $u(0) = u(1)$ and $u'(0) = u'(1)$ cannot be solved. This corresponds to the singular circulant matrix C_n.

Question 4. (10 pts.) Orthonormal eigenvectors.

We saw in class that matrix K is symmetric and thus has a full set of orthogonal eigenvectors. We like to have orthogonal eigenvectors because of the following, which we show for K_2 but holds in general.

a) We know the eigenvalues of K_2 are $\lambda_1 = 1$ and $\lambda_2 = 3$. Find the eigenvectors for those eigenvalues, and make them unit vectors. That is, find $\vec{q}_1$ and $\vec{q}_2$ such that $K_2\vec{q}_j = \lambda_j\vec{q}_j$ and $\vec{q}_j^T\vec{q}_j = 1$ for $j = 1, 2$.
b) Verify that the eigenvectors $\vec{q}_1$ and $\vec{q}_2$ are orthonormal (orthogonal and of unit length).
c) Verify by hand that $K_2 = Q\Lambda Q^T$ where Λ is the matrix with the eigenvalues on its diagonal, and Q is the matrix with columns $\vec{q}_1$ and $\vec{q}_2$.
   This is not a coincidence! When a matrix has orthogonal eigenvectors, as do symmetric matrices, we can decompose it as a product of the matrix of orthonormalized eigenvectors multiplied by the matrix of eigenvalues, multiplied by the *transpose* (and not the *inverse* as usual) of the matrix of orthonormalized eigenvectors.
d) Verify by hand that $Q^TQ = I$ and $QQ^T = I$ as well, where I is the 2 by 2 identity matrix. This confirms that $Q^T = Q^{-1}$!

We will see in class that this leads to easier inverses of matrices, and also to faster multiplication of powers of matrices.

Question 5. (10 pts.) From eigenvectors to eigenfunctions.

We saw in pset1 Q2 that $-\frac{d^2}{dx^2}\sin(\pi x) = \pi^2\sin(\pi x)$ with fixed-fixed boundary conditions... It looks as if π^2 is an "eigenvalue" of the operator $-\frac{d^2}{dx^2}$, and that $\sin(\pi x)$ is the corresponding "eigenvector". More generally, we have $-\frac{d^2}{dx^2}\sin(k\pi x) = k^2\pi^2\sin(k\pi x)$ for non-zero integer k.

a) Construct $K = K_6$ in Matlab, and type "`[Q E]=eig((1/h^2)*K);`". This gives you the eigenvectors as columns of Q, and eigenvalues in the diagonal of E. What is h, and n? Confirm that eigenvalues are $(2 - 2\cos(k\pi h))/h^2$, for $k = 1, 2, \ldots, n$.
b) Show that, in the limit as $h \to 0$, we recover the eigenvalues we expect from the discussion above, that is:

$$\lim_{h\to} \frac{1}{h^2}(2 - 2\cos(k\pi h)) = k^2\pi^2.$$

   Hint: you may use the following approximation for the cosine, from the Taylor series: $cos(x) \approx 1 - x^2/2$.
c) We would expect sines to appear in the eigenvectors of K_6, again because of the discussion above. In particular, we expect the functions $\sin(k\pi x)$, discretized to $\sin(k\pi jh)$ for $j = 1, 2, \ldots, n$. Those give us the *discrete sine transform* (TYPO IN NEXT EXPRESSION, WAS MISSING FACTOR OF $\sqrt{2}$):
   "`DST=sin(JK*pi*h)*sqrt(2)/sqrt(n+1)`",
   where matrix "`JK=[1:6]'*[1:6]`" (TYPO IN PREVIOUS EXPRESSION, I HAD UP TO 5 BUT IT SHOULD BE UP TO 6) has entries j times k. Does the eigenvector matrix Q correspond to the DST? If not, what is the difference?

d) Since K_6 is symmetric, we expect orthogonal eigenvectors. Verify in Matlab that $Q^{-1} = Q^T$. Verify as well that $\text{DST}^T = \text{DST}^{-1}$. This is a crucial property of the DST!

Question 6. (15 pts.) Linear algebra and differential equations practice.

a) In general, if a matrix has a zero eigenvalue, what does this mean about the invertibility of the matrix? Justify your answer.

b) The matrix B (free-free boundary conditions), which is K but with top left and bottom right entries of 1, has an eigenvector with 0 eigenvalue. What is this eigenvector? What does your answer to a) tell you about the invertibility of B?

c) Let A be an n by n matrix with n independent eigenvectors. Let S be its eigenvector matrix. That is, the n columns of S are the eigenvectors of A, and we may write $A = S\lambda S^{-1}$, with eigenvalues of A in the diagonal of Λ. What are the eigenvalues of $A - 3I$? Its eigenvectors?

d) Find the eigenvalues λ_1 and λ_2 and eigenvectors $\vec{y}_1$ and $\vec{y}_2$ of

$$A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$$

Use those to produce solutions $\vec{u}(t) = e^{\lambda_j t}\vec{y}_j$ to $d\vec{u}/dt = A\vec{u}$. Show those are indeed solutions.