18.085, PROBLEM SET 1, DUE 6/13 NOON IN E17-301Q

Question 1. (20 pts.) This exercise gets us started on using finite differences.

Please download file "`pset1_1.m`" from the course website. This solves the following problem (fixed-fixed ends):

$$-u'' = 1, \ x \in [0,1], \ u(0) = 0, \ u(1) = 0$$

Open it in Matlab, and try out the commands to see what they do. Then, run the file (type "`pset1_1`" in the command line, from the folder that contains the file). Try changing the value of n in the file, saving it and running it again, to see what diferent values of the error e you obtain. Describe what happens to error e for different values of n, and why that is the case.

Question 2. (20 pts.) Finite differences, a step up.

We saw in class that finite differences happen to be exact on quadratics. What if we do not have a quadratic? Consider now a function $u(x) = \sin(\pi x)$.

a) What boundary conditions does the solution u satisfy at $x = 0$ and $x = 1$?
b) What is the negative of the second derivative in x of u? Call it $f(x)$. (Hence $f = -u''$.)
c) Use your answers to a and b to modify file "`pset1_1.m`" into a new file, "`pset1_2.m`", in order to solve the equation $-u'' = f$ with f as in b). Submit a print-out of that new file "`pset1_2.m`" with your pset solution.
d) Again, try a few different values of n, and describe what happens to the error e. Why do we expect that behavior? Be as precise as you can in your explanation.

Question 3. (20 pts.) This exercise and the next will confirm that treating a free boundary condition correctly does give better accuracy.

Recall we found two ways of dealing with the free boundary condition on the left side. The first is only accurate to first order, and we begin with that. You will modify file "`pset1_1.m`" into "`pset1_3.m`". Print your file "`pset1_3.m`" and submit it with your pset solution. We wish to solve the differential equation $-u'' = 1$ as in problem 1, but with free-fixed ends (that is, $u'(0) = 0$ and $u(1) = 0$).

a) We will consider the error at test point $x = 0$ now, so change that accordingly in your file, along with the new true solution ut as we found in class.
b) Now the crucial part: change K to T! So define T correctly first, then find the solution using T and the backslash algorithm.
c) Almost done! Now remember we used previously in "`pset1_1.m`" the command "`u(round(xtest/h))`" to pick the correct entry of the vector u. That might cause you trouble now. Fix it. (Think of what we know is equal to u_0.)

   (This should be a lesson: the slightest modification in a code might require you to carefully comb through the rest of your code. That is, until they figure out how brains can directly communicate with computers.)
d) Look at the error e at test point $xtest = 0$. What happens as you increase n? Why?

Question 4. (20 pts.) Now we treat the free boundary with higher accuracy!

You will modify file "`pset1_3.m`" into "`pset1_4.m`". Print your file "`pset1_4.m`" and submit it with your pset solution.

a) Think of the unknowns we have, and how many. Change vector f if needed in your file to account for that.
b) You should have already changed K to T in problem 3. However, if the number of unknown has changed, so shoudl the dimensions of T.
c) Recall that with the second-order accurate formulation of the boundary condition, we divided the top row by 2. Modify vector f accordingly.
d) Almost done! Now remember we used previously in "`pset1_1.m`" the command "`u(round(xtest/h))`" to pick the correct entry of the vector u. But since our unknowns may have changed, maybe this has changed too. Fix it if needed.
e) Look at the error e at test point $xtest = 0$. What happens as you increase n? Why?

The following questions test your understanding of linear algebra.

Question 5. (10 pts.) Elimination practice.

a) (From the textbook.) Carry out by hand the elimination of the circulant matrix C_4 to reach an upper triangular U and lower triangular L. You may confirm your result using Matlab, but please do it by hand as well and show your steps. You may also confirm your result (and test your matrix multiplication skills) by multiplying L and U together (optional, but good to keep in mind for the exam).
b) Complete: The last pivot of U is ______ because the matrix C is ______ .
c) The last column of U has new non-zeros. Explain why this "fill-in" occurs.

Question 6. (10 pts.) Multiplication practice.

(From the textbook.) You can multiply $A\vec{x}$ by rows (usual way) or by columns (more important). Small example:

By rows: $$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 20 \end{pmatrix} = \begin{pmatrix} \text{inner product using row 1} \\ \text{inner product using row 2} \end{pmatrix} = \begin{pmatrix} 2 \times 10 + 3 \times 20 \\ 4 \times 10 + 5 \times 20 \end{pmatrix} = \begin{pmatrix} 80 \\ 140 \end{pmatrix}$$

By columns: $$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 20 \end{pmatrix} = \text{combination of columns} = 10 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 20 \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 80 \\ 140 \end{pmatrix}$$

Now, try these two ways yourself, showing your work as I did in the example. For this, use the matrix B_3 and the all-ones vector. You know what the result should be!